

RD Sharma Solutions for Class 11 Maths Chapter 8 Transformation Formulae

1. Express each of the following as the sum or difference of sines and cosines:

(i) $2 \sin 3x \cos x$

(ii) $2 \cos 3x \sin 2x$

(iii) $2 \sin 4x \sin 3x$

(iv) $2 \cos 7x \cos 3x$

Solution:

(i) $2 \sin 3x \cos x$

By using the formula,

$$2 \sin A \cos B = \sin (A + B) + \sin (A - B)$$

$$2 \sin 3x \cos x = \sin (3x + x) + \sin (3x - x)$$

$$= \sin (4x) + \sin (2x)$$

$$= \sin 4x + \sin 2x$$

(ii) $2 \cos 3x \sin 2x$

By using the formula,

$$2 \cos A \sin B = \sin (A + B) - \sin (A - B)$$

$$2 \cos 3x \sin 2x = \sin (3x + 2x) - \sin (3x - 2x)$$

$$= \sin (5x) - \sin (x)$$

$$= \sin 5x - \sin x$$

(iii) $2 \sin 4x \sin 3x$

By using the formula,

$$2 \sin A \sin B = \cos (A - B) - \cos (A + B)$$

$$2 \sin 4x \sin 3x = \cos (4x - 3x) - \cos (4x + 3x)$$

$$= \cos (x) - \cos (7x)$$

$$= \cos x - \cos 7x$$

(iv) $2 \cos 7x \cos 3x$

By using the formula,

$$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$$

$$2 \cos 7x \cos 3x = \cos (7x + 3x) + \cos (7x - 3x)$$

$$= \cos (10x) + \cos (4x)$$

$$= \cos 10x + \cos 4x$$

2. Prove that:

(i) $2 \sin 5\pi/12 \sin \pi/12 = 1/2$

(ii) $2 \cos 5\pi/12 \cos \pi/12 = 1/2$

(iii) $2 \sin 5\pi/12 \cos \pi/12 = (\sqrt{3} + 2)/2$

Solution:

(i) $2 \sin 5\pi/12 \sin \pi/12 = 1/2$

By using the formula,

$$2 \sin A \sin B = \cos (A - B) - \cos (A + B)$$

$$2 \sin 5\pi/12 \sin \pi/12 = \cos (5\pi/12 - \pi/12) - \cos (5\pi/12 + \pi/12)$$

$$= \cos (4\pi/12) - \cos (6\pi/12)$$

$$= \cos (\pi/3) - \cos (\pi/2)$$

$$= \cos (180^\circ/3) - \cos (180^\circ/2)$$

$$= \cos 60^\circ - \cos 90^\circ$$

$$= 1/2 - 0$$

$$= 1/2$$

Hence Proved.

(ii) $2 \cos 5\pi/12 \cos \pi/12 = 1/2$

By using the formula,

$$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$$

$$2 \cos 5\pi/12 \cos \pi/12 = \cos (5\pi/12 + \pi/12) + \cos (5\pi/12 - \pi/12)$$

$$= \cos (6\pi/12) + \cos (4\pi/12)$$

$$= \cos (\pi/2) + \cos (\pi/3)$$

$$= \cos (180^\circ/2) + \cos (180^\circ/3)$$

$$= \cos 90^\circ + \cos 60^\circ$$

$$= 0 + 1/2$$

$$= 1/2$$

Hence Proved.

(iii) $2 \sin 5\pi/12 \cos \pi/12 = (\sqrt{3} + 2)/2$

By using the formula,

$$2 \sin A \cos B = \sin (A + B) + \sin (A - B)$$

$$2 \sin 5\pi/12 \cos \pi/12 = \sin (5\pi/12 + \pi/12) + \sin (5\pi/12 - \pi/12)$$

$$= \sin (6\pi/12) + \sin (4\pi/12)$$

$$= \sin (\pi/2) + \sin (\pi/3)$$

$$= \sin (180^\circ/2) + \sin (180^\circ/3)$$

$$= \sin 90^\circ + \sin 60^\circ$$

$$= 1 + \sqrt{3}$$

$$= (2 + \sqrt{3})/2$$

$$= (\sqrt{3} + 2)/2$$

Hence Proved.

3. show that:

$$(i) \sin 50^\circ \cos 85^\circ = (1 - \sqrt{2}\sin 35^\circ)/2\sqrt{2}$$

$$(ii) \sin 25^\circ \cos 115^\circ = 1/2 \{\sin 140^\circ - 1\}$$

Solution:

$$(i) \sin 50^\circ \cos 85^\circ = (1 - \sqrt{2}\sin 35^\circ)/2\sqrt{2}$$

By using the formula,

$$2 \sin A \cos B = \sin (A + B) + \sin (A - B)$$

$$\sin A \cos B = [\sin (A + B) + \sin (A - B)] / 2$$

$$\sin 50^\circ \cos 85^\circ = [\sin(50^\circ + 85^\circ) + \sin (50^\circ - 85^\circ)] / 2$$

$$= [\sin (135^\circ) + \sin (-35^\circ)] / 2$$

$$= [\sin (135^\circ) - \sin (35^\circ)] / 2 \text{ (since, } \sin (-x) = -\sin x)$$

$$= [\sin (180^\circ - 45^\circ) - \sin 35^\circ] / 2$$

$$= [\sin 45^\circ - \sin 35^\circ] / 2$$

$$= [(1/\sqrt{2}) - \sin 35^\circ] / 2$$

$$= [(1 - \sin 35^\circ)/\sqrt{2}] / 2$$

$$= (1 - \sin 35^\circ) / 2\sqrt{2}$$

Hence proved.

$$(ii) \sin 25^\circ \cos 115^\circ = 1/2 \{\sin 140^\circ - 1\}$$

By using the formula,

$$2 \sin A \cos B = \sin (A + B) + \sin (A - B)$$

$$\sin A \cos B = [\sin (A + B) + \sin (A - B)] / 2$$

$$\sin 20^\circ \cos 115^\circ = [\sin(25^\circ + 115^\circ) + \sin (25^\circ - 115^\circ)] / 2$$

$$= [\sin (140^\circ) + \sin (-90^\circ)] / 2$$

$$= [\sin (140^\circ) - \sin (90^\circ)] / 2 \text{ (since, } \sin (-x) = -\sin x)$$

$$= 1/2 \{\sin 140^\circ - 1\}$$

Hence proved.

4. Prove that:

$$4 \cos x \cos (\pi/3 + x) \cos (\pi/3 - x) = \cos 3x$$

Solution:

Let us consider LHS:

$$4 \cos x \cos (\pi/3 + x) \cos (\pi/3 - x) = 2 \cos x (2 \cos (\pi/3 + x) \cos (\pi/3 - x))$$

By using the formula,

$$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$$

$$2 \cos x (2 \cos (\pi/3 + x) \cos (\pi/3 - x)) = 2 \cos x (\cos (\pi/3 + x + \pi/3 - x) + \cos (\pi/3 + x - \pi/3 - x))$$

$$= 2 \cos x (\cos (2\pi/3) + \cos (2x))$$

$$= 2 \cos x \{\cos 120^\circ + \cos 2x\}$$

$$= 2 \cos x \{\cos (180^\circ - 60^\circ) + \cos 2x\}$$

$$= 2 \cos x (\cos 2x - \cos 60^\circ) \text{ (since, } \{\cos (180^\circ - A) = -\cos A\})$$

$$= 2 \cos 2x \cos x - 2 \cos x \cos 60^\circ$$

$$= (\cos (x + 2x) + \cos (2x - x)) - (2 \cos x)/2$$

$$= \cos 3x + \cos x - \cos x$$

$$= \cos 3x$$

$$= \text{RHS}$$

Hence Proved.

EXERCISE 8.2 PAGE NO: 8.17

1. Express each of the following as the product of sines and cosines:

(i) $\sin 12x + \sin 4x$

(ii) $\sin 5x - \sin x$

(iii) $\cos 12x + \cos 8x$

(iv) $\cos 12x - \cos 4x$

(v) $\sin 2x + \cos 4x$

Solution:

(i) $\sin 12x + \sin 4x$

By using the formula,

$$\sin A + \sin B = 2 \sin (A+B)/2 \cos (A-B)/2$$

$$\sin 12x + \sin 4x = 2 \sin (12x + 4x)/2 \cos (12x - 4x)/2$$

$$= 2 \sin 16x/2 \cos 8x/2$$

$$= 2 \sin 8x \cos 4x$$

(ii) $\sin 5x - \sin x$

By using the formula,

$$\sin A - \sin B = 2 \cos (A+B)/2 \sin (A-B)/2$$

$$\sin 5x - \sin x = 2 \cos (5x + x)/2 \sin (5x - x)/2$$

$$= 2 \cos 6x/2 \sin 4x/2$$

$$= 2 \cos 3x \sin 2x$$

$$\text{(iii) } \cos 12x + \cos 8x$$

By using the formula,

$$\cos A + \cos B = 2 \cos (A+B)/2 \cos (A-B)/2$$

$$\cos 12x + \cos 8x = 2 \cos (12x + 8x)/2 \cos (12x - 8x)/2$$

$$= 2 \cos 20x/2 \cos 4x/2$$

$$= 2 \cos 10x \cos 2x$$

$$\text{(iv) } \cos 12x - \cos 4x$$

By using the formula,

$$\cos A - \cos B = -2 \sin (A+B)/2 \sin (A-B)/2$$

$$\cos 12x - \cos 4x = -2 \sin (12x + 4x)/2 \sin (12x - 4x)/2$$

$$= -2 \sin 16x/2 \sin 8x/2$$

$$= -2 \sin 8x \sin 4x$$

$$\text{(v) } \sin 2x + \cos 4x$$

$$\sin 2x + \cos 4x = \sin 2x + \sin (90^\circ - 4x)$$

By using the formula,

$$\sin A + \sin B = 2 \sin (A+B)/2 \cos (A-B)/2$$

$$\sin 2x + \sin (90^\circ - 4x) = 2 \sin (2x + 90^\circ - 4x)/2 \cos (2x - 90^\circ + 4x)/2$$

$$= 2 \sin (90^\circ - 2x)/2 \cos (6x - 90^\circ)/2$$

$$= 2 \sin (45^\circ - x) \cos (3x - 45^\circ)$$

$$= 2 \sin (45^\circ - x) \cos \{-(45^\circ - 3x)\} \text{ (since, } \{\cos (-x) = \cos x\})$$

$$= 2 \sin (45^\circ - x) \cos (45^\circ - 3x)$$

$$= 2 \sin (\pi/4 - x) \cos (\pi/4 - 3x)$$

2. Prove that :

$$\text{(i) } \sin 38^\circ + \sin 22^\circ = \sin 82^\circ$$

$$\text{(ii) } \cos 100^\circ + \cos 20^\circ = \cos 40^\circ$$

$$\text{(iii) } \sin 50^\circ + \sin 10^\circ = \cos 20^\circ$$

$$\text{(iv) } \sin 23^\circ + \sin 37^\circ = \cos 7^\circ$$

$$\text{(v) } \sin 105^\circ + \cos 105^\circ = \cos 45^\circ$$

$$\text{(vi) } \sin 40^\circ + \sin 20^\circ = \cos 10^\circ$$

Solution:

$$\text{(i) } \sin 38^\circ + \sin 22^\circ = \sin 82^\circ$$

Let us consider LHS:

$$\sin 38^\circ + \sin 22^\circ$$

By using the formula,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\sin 38^\circ + \sin 22^\circ = 2 \sin \frac{(38^\circ + 22^\circ)}{2} \cos \frac{(38^\circ - 22^\circ)}{2}$$

$$= 2 \sin 60^\circ/2 \cos 16^\circ/2$$

$$= 2 \sin 30^\circ \cos 8^\circ$$

$$= 2 \times \frac{1}{2} \times \cos 8^\circ$$

$$= \cos 8^\circ$$

$$= \cos (90^\circ - 82^\circ)$$

$$= \sin 82^\circ \text{ (since, } \{\cos (90^\circ - A) = \sin A\})$$

$$= \text{RHS}$$

Hence Proved.

$$\text{(ii) } \cos 100^\circ + \cos 20^\circ = \cos 40^\circ$$

Let us consider LHS:

$$\cos 100^\circ + \cos 20^\circ$$

By using the formula,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\cos 100^\circ + \cos 20^\circ = 2 \cos \frac{(100^\circ + 20^\circ)}{2} \cos \frac{(100^\circ - 20^\circ)}{2}$$

$$= 2 \cos 120^\circ/2 \cos 80^\circ/2$$

$$= 2 \cos 60^\circ \cos 4^\circ$$

$$= 2 \times \frac{1}{2} \times \cos 40^\circ$$

$$= \cos 40^\circ$$

$$= \text{RHS}$$

Hence Proved.

$$\text{(iii) } \sin 50^\circ + \sin 10^\circ = \cos 20^\circ$$

Let us consider LHS:

$$\sin 50^\circ + \sin 10^\circ$$

By using the formula,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\sin 50^\circ + \sin 10^\circ = 2 \sin \frac{(50^\circ + 10^\circ)}{2} \cos \frac{(50^\circ - 10^\circ)}{2}$$

$$= 2 \sin 60^\circ/2 \cos 40^\circ/2$$

$$= 2 \sin 30^\circ \cos 20^\circ$$

$$= 2 \times \frac{1}{2} \times \cos 20^\circ$$

$$= \cos 20^\circ$$

$$= \text{RHS}$$

Hence Proved.

$$\text{(iv)} \sin 23^\circ + \sin 37^\circ = \cos 7^\circ$$

Let us consider LHS:

$$\sin 23^\circ + \sin 37^\circ$$

By using the formula,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\sin 23^\circ + \sin 37^\circ = 2 \sin \frac{(23^\circ + 37^\circ)}{2} \cos \frac{(23^\circ - 37^\circ)}{2}$$

$$= 2 \sin 60^\circ/2 \cos -14^\circ/2$$

$$= 2 \sin 30^\circ \cos -7^\circ$$

$$= 2 \times \frac{1}{2} \times \cos -7^\circ$$

$$= \cos 7^\circ \text{ (since, } \{\cos (-A) = \cos A\})$$

$$= \text{RHS}$$

Hence Proved.

$$\text{(v)} \sin 105^\circ + \cos 105^\circ = \cos 45^\circ$$

Let us consider LHS: $\sin 105^\circ + \cos 105^\circ$

$$\sin 105^\circ + \cos 105^\circ = \sin 105^\circ + \sin (90^\circ - 105^\circ) \text{ [since, } \{\sin (90^\circ - A) = \cos A\}]$$

$$= \sin 105^\circ + \sin (-15^\circ)$$

$$= \sin 105^\circ - \sin 15^\circ \text{ [}\{\sin(-A) = -\sin A\}\text{]}$$

By using the formula,

$$\sin A - \sin B = 2 \cos \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

$$\sin 105^\circ - \sin 15^\circ = 2 \cos \frac{(105^\circ + 15^\circ)}{2} \sin \frac{(105^\circ - 15^\circ)}{2}$$

$$= 2 \cos 120^\circ/2 \sin 90^\circ/2$$

$$= 2 \cos 60^\circ \sin 45^\circ$$

$$= 2 \times \frac{1}{2} \times \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}$$

$$= \cos 45^\circ$$

$$= \text{RHS}$$

Hence proved.

$$\text{(vi)} \sin 40^\circ + \sin 20^\circ = \cos 10^\circ$$

Let us consider LHS:

$$\sin 40^\circ + \sin 20^\circ$$

By using the formula,

$$\sin A + \sin B = 2 \sin (A+B)/2 \cos (A-B)/2$$

$$\sin 40^\circ + \sin 20^\circ = 2 \sin (40^\circ + 20^\circ)/2 \cos (40^\circ - 20^\circ)/2$$

$$= 2 \sin 60^\circ/2 \cos 20^\circ/2$$

$$= 2 \sin 30^\circ \cos 10^\circ$$

$$= 2 \times 1/2 \times \cos 10^\circ$$

$$= \cos 10^\circ$$

$$= \text{RHS}$$

Hence Proved.

3. Prove that:

$$(i) \cos 55^\circ + \cos 65^\circ + \cos 175^\circ = 0$$

$$(ii) \sin 50^\circ - \sin 70^\circ + \sin 10^\circ = 0$$

$$(iii) \cos 80^\circ + \cos 40^\circ - \cos 20^\circ = 0$$

$$(iv) \cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$$

$$(v) \sin 5\pi/18 - \cos 4\pi/9 = \sqrt{3} \sin \pi/9$$

$$(vi) \cos \pi/12 - \sin \pi/12 = 1/\sqrt{2}$$

$$(vii) \sin 80^\circ - \cos 70^\circ = \cos 50^\circ$$

$$(viii) \sin 51^\circ + \cos 81^\circ = \cos 21^\circ$$

Solution:

$$(i) \cos 55^\circ + \cos 65^\circ + \cos 175^\circ = 0$$

Let us consider LHS:

$$\cos 55^\circ + \cos 65^\circ + \cos 175^\circ$$

By using the formula,

$$\cos A + \cos B = 2 \cos (A+B)/2 \cos (A-B)/2$$

$$\cos 55^\circ + \cos 65^\circ + \cos 175^\circ = 2 \cos (55^\circ + 65^\circ)/2 \cos (55^\circ - 65^\circ) + \cos (180^\circ - 5^\circ)$$

$$= 2 \cos 120^\circ/2 \cos (-10^\circ)/2 - \cos 5^\circ \text{ (since, } \{\cos (180^\circ - A) = -\cos A\})$$

$$= 2 \cos 60^\circ \cos (-5^\circ) - \cos 5^\circ \text{ (since, } \{\cos (-A) = \cos A\})$$

$$= 2 \times 1/2 \times \cos 5^\circ - \cos 5^\circ$$

$$= 0$$

$$= \text{RHS}$$

Hence Proved.

$$(ii) \sin 50^\circ - \sin 70^\circ + \sin 10^\circ = 0$$

Let us consider LHS:

$$\sin 50^\circ - \sin 70^\circ + \sin 10^\circ$$

By using the formula,

$$\sin A - \sin B = 2 \cos (A+B)/2 \sin (A-B)/2$$

$$\sin 50^\circ - \sin 70^\circ + \sin 10^\circ = 2 \cos (50^\circ + 70^\circ)/2 \sin (50^\circ - 70^\circ) + \sin 10^\circ$$

$$= 2 \cos 120^\circ/2 \sin (-20^\circ)/2 + \sin 10^\circ$$

$$= 2 \cos 60^\circ (-\sin 10^\circ) + \sin 10^\circ \text{ [since, } \{\sin (-A) = -\sin (A)\}]$$

$$= 2 \times 1/2 \times -\sin 10^\circ + \sin 10^\circ$$

$$= 0$$

$$= \text{RHS}$$

Hence proved.

$$\text{(iii) } \cos 80^\circ + \cos 40^\circ - \cos 20^\circ = 0$$

Let us consider LHS:

$$\cos 80^\circ + \cos 40^\circ - \cos 20^\circ$$

By using the formula,

$$\cos A + \cos B = 2 \cos (A+B)/2 \cos (A-B)/2$$

$$\cos 80^\circ + \cos 40^\circ - \cos 20^\circ = 2 \cos (80^\circ + 40^\circ)/2 \cos (80^\circ - 40^\circ) - \cos 20^\circ$$

$$= 2 \cos 120^\circ/2 \cos 40^\circ/2 - \cos 20^\circ$$

$$= 2 \cos 60^\circ \cos 20^\circ - \cos 20^\circ$$

$$= 2 \times 1/2 \times \cos 20^\circ - \cos 20^\circ$$

$$= 0$$

$$= \text{RHS}$$

Hence Proved.

$$\text{(iv) } \cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$$

Let us consider LHS:

$$\cos 20^\circ + \cos 100^\circ + \cos 140^\circ$$

By using the formula,

$$\cos A + \cos B = 2 \cos (A+B)/2 \cos (A-B)/2$$

$$\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 2 \cos (20^\circ + 100^\circ)/2 \cos (20^\circ - 100^\circ) + \cos (180^\circ - 40^\circ)$$

$$= 2 \cos 120^\circ/2 \cos (-80^\circ)/2 - \cos 40^\circ \text{ (since, } \{\cos (180^\circ - A) = -\cos A\})$$

$$= 2 \cos 60^\circ \cos (-40^\circ) - \cos 40^\circ \text{ (since, } \{\cos (-A) = \cos A\})$$

$$= 2 \times 1/2 \times \cos 40^\circ - \cos 40^\circ$$

$$= 0$$

= RHS

Hence Proved.

$$(v) \sin 5\pi/18 - \cos 4\pi/9 = \sqrt{3} \sin \pi/9$$

Let us consider LHS:

$$\sin 5\pi/18 - \cos 4\pi/9 = \sin 5\pi/18 - \sin (\pi/2 - 4\pi/9) \text{ (since, } \cos A = \sin (90^\circ - A))$$

$$= \sin 5\pi/18 - \sin (9\pi - 8\pi)/18$$

$$= \sin 5\pi/18 - \sin \pi/18$$

By using the formula,

$$\sin A - \sin B = 2 \cos (A+B)/2 \sin (A-B)/2$$

$$\sin \frac{5\pi}{18} - \sin \frac{\pi}{18} = 2 \cos \left(\frac{\frac{5\pi}{18} + \frac{\pi}{18}}{2} \right) \sin \left(\frac{\frac{5\pi}{18} - \frac{\pi}{18}}{2} \right)$$

$$= 2 \cos (6\pi/36) \sin (4\pi/36)$$

$$= 2 \cos \pi/6 \sin \pi/9$$

$$= 2 \cos 30^\circ \sin \pi/9$$

$$= 2 \times \sqrt{3}/2 \times \sin \pi/9$$

$$= \sqrt{3} \sin \pi/9$$

= RHS

Hence proved.

$$(vi) \cos \pi/12 - \sin \pi/12 = 1/\sqrt{2}$$

Let us consider LHS:

$$\cos \pi/12 - \sin \pi/12 = \sin (\pi/2 - \pi/12) - \sin \pi/12 \text{ (since, } \cos A = \sin(90^\circ - A))$$

$$= \sin (6\pi - 5\pi)/12 - \sin \pi/12$$

$$= \sin 5\pi/12 - \sin \pi/12$$

By using the formula,

$$\sin A - \sin B = 2 \cos (A+B)/2 \sin (A-B)/2$$

$$\sin \frac{5\pi}{12} - \sin \frac{\pi}{12} = 2 \cos \left(\frac{\frac{5\pi}{12} + \frac{\pi}{12}}{2} \right) \sin \left(\frac{\frac{5\pi}{12} - \frac{\pi}{12}}{2} \right)$$

$$= 2 \cos (6\pi/24) \sin (4\pi/24)$$

$$= 2 \cos \pi/4 \sin \pi/6$$

$$= 2 \cos 45^\circ \sin 30^\circ$$

$$= 2 \times 1/\sqrt{2} \times 1/2$$

$$= 1/\sqrt{2}$$

= RHS

Hence proved.

(vii) $\sin 80^\circ - \cos 70^\circ = \cos 50^\circ$

$$\sin 80^\circ = \cos 50^\circ + \cos 70^\circ$$

So, now let us consider RHS

$$\cos 50^\circ + \cos 70^\circ$$

By using the formula,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\cos 50^\circ + \cos 70^\circ = 2 \cos \frac{(50^\circ + 70^\circ)}{2} \cos \frac{(50^\circ - 70^\circ)}{2}$$

$$= 2 \cos 120^\circ/2 \cos (-20^\circ)/2$$

$$= 2 \cos 60^\circ \cos (-10^\circ)$$

$$= 2 \times \frac{1}{2} \times \cos 10^\circ \text{ (since, } \cos(-A) = \cos A \text{)}$$

$$= \cos 10^\circ$$

$$= \cos (90^\circ - 80^\circ)$$

$$= \sin 80^\circ \text{ (since, } \cos (90^\circ - A) = \sin A \text{)}$$

$$= \text{LHS}$$

Hence Proved.

(viii) $\sin 51^\circ + \cos 81^\circ = \cos 21^\circ$

Let us consider LHS:

$$\sin 51^\circ + \cos 81^\circ = \sin 51^\circ + \sin (90^\circ - 81^\circ)$$

$$= \sin 51^\circ + \sin 9^\circ \text{ (since, } \sin (90^\circ - A) = \cos A \text{)}$$

By using the formula,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\sin 51^\circ + \sin 9^\circ = 2 \sin \frac{(51^\circ + 9^\circ)}{2} \cos \frac{(51^\circ - 9^\circ)}{2}$$

$$= 2 \sin 60^\circ/2 \cos 42^\circ/2$$

$$= 2 \sin 30^\circ \cos 21^\circ$$

$$= 2 \times \frac{1}{2} \times \cos 21^\circ$$

$$= \cos 21^\circ$$

$$= \text{RHS}$$

Hence proved.

4. Prove that:

(i) $\cos \left(\frac{3\pi}{4} + x\right) - \cos \left(\frac{3\pi}{4} - x\right) = -\sqrt{2} \sin x$

(ii) $\cos \left(\frac{\pi}{4} + x\right) + \cos \left(\frac{\pi}{4} - x\right) = \sqrt{2} \cos x$

Solution:

$$(i) \cos (3\pi/4 + x) - \cos (3\pi/4 - x) = -\sqrt{2} \sin x$$

Let us consider LHS:

$$\cos (3\pi/4 + x) - \cos (3\pi/4 - x)$$

By using the formula,

$$\cos A - \cos B = -2 \sin (A+B)/2 \sin (A-B)/2$$

$$\cos (3\pi/4 + x) - \cos (3\pi/4 - x) = -2 \sin (3\pi/4 + x + 3\pi/4 - x)/2 \sin (3\pi/4 + x - 3\pi/4 - x)/2$$

$$= -2 \sin (6\pi/4)/2 \sin 2x/2$$

$$= -2 \sin 6\pi/8 \sin x$$

$$= -2 \sin 3\pi/4 \sin x$$

$$= -2 \sin (\pi - \pi/4) \sin x$$

$$= -2 \sin \pi/4 \sin x \text{ (since, } (\pi - A) = \sin A)$$

$$= -2 \times 1/\sqrt{2} \times \sin x$$

$$= -\sqrt{2} \sin x$$

$$= \text{RHS}$$

Hence proved.

$$(ii) \cos (\pi/4 + x) + \cos (\pi/4 - x) = \sqrt{2} \cos x$$

Let us consider LHS:

$$\cos (\pi/4 + x) + \cos (\pi/4 - x)$$

By using the formula,

$$\cos A + \cos B = 2 \cos (A+B)/2 \cos (A-B)/2$$

$$\cos (\pi/4 + x) + \cos (\pi/4 - x) = 2 \cos (\pi/4 + x + \pi/4 - x)/2 \cos (\pi/4 + x - \pi/4 - x)/2$$

$$= 2 \cos (2\pi/4)/2 \cos 2x/2$$

$$= 2 \cos 2\pi/8 \cos x$$

$$= 2 \sin \pi/4 \cos x$$

$$= 2 \times 1/\sqrt{2} \times \cos x$$

$$= \sqrt{2} \cos x$$

$$= \text{RHS}$$

Hence proved.

5. Prove that:

$$(i) \sin 65^\circ + \cos 65^\circ = \sqrt{2} \cos 20^\circ$$

$$(ii) \sin 47^\circ + \cos 77^\circ = \cos 17^\circ$$

Solution:

$$(i) \sin 65^\circ + \cos 65^\circ = \sqrt{2} \cos 20^\circ$$

Let us consider LHS:

$$\sin 65^\circ + \cos 65^\circ = \sin 65^\circ + \sin (90^\circ - 65^\circ)$$

$$= \sin 65^\circ + \sin 25^\circ$$

By using the formula,

$$\sin A + \sin B = 2 \sin (A+B)/2 \cos (A-B)/2$$

$$\sin 65^\circ + \sin 25^\circ = 2 \sin (65^\circ + 25^\circ)/2 \cos (65^\circ - 25^\circ)/2$$

$$= 2 \sin 90^\circ/2 \cos 40^\circ/2$$

$$= 2 \sin 45^\circ \cos 20^\circ$$

$$= 2 \times 1/\sqrt{2} \times \cos 20^\circ$$

$$= \sqrt{2} \cos 20^\circ$$

$$= \text{RHS}$$

Hence proved.

$$(ii) \sin 47^\circ + \cos 77^\circ = \cos 17^\circ$$

Let us consider LHS:

$$\sin 47^\circ + \cos 77^\circ = \sin 47^\circ + \sin (90^\circ - 77^\circ)$$

$$= \sin 47^\circ + \sin 13^\circ$$

By using the formula,

$$\sin A + \sin B = 2 \sin (A+B)/2 \cos (A-B)/2$$

$$\sin 47^\circ + \sin 13^\circ = 2 \sin (47^\circ + 13^\circ)/2 \cos (47^\circ - 13^\circ)/2$$

$$= 2 \sin 60^\circ/2 \cos 34^\circ/2$$

$$= 2 \sin 30^\circ \cos 17^\circ$$

$$= 2 \times 1/2 \times \cos 17^\circ$$

$$= \cos 17^\circ$$

$$= \text{RHS}$$

Hence proved.

6. Prove that:

$$(i) \cos 3A + \cos 5A + \cos 7A + \cos 15A = 4 \cos 4A \cos 5A \cos 6A$$

$$(ii) \cos A + \cos 3A + \cos 5A + \cos 7A = 4 \cos A \cos 2A \cos 4A$$

$$(iii) \sin A + \sin 2A + \sin 4A + \sin 5A = 4 \cos A/2 \cos 3A/2 \sin 3A$$

$$(iv) \sin 3A + \sin 2A - \sin A = 4 \sin A \cos A/2 \cos 3A/2$$

$$(v) \cos 20^\circ \cos 100^\circ + \cos 100^\circ \cos 140^\circ - \cos 140^\circ \cos 200^\circ = -3/4$$

$$(vi) \sin x/2 \sin 7x/2 + \sin 3x/2 \sin 11x/2 = \sin 2x \sin 5x$$

(vii) $\cos x \cos x/2 - \cos 3x \cos 9x/2 = \sin 4x \sin 7x/2$

Solution:

(i) $\cos 3A + \cos 5A + \cos 7A + \cos 15A = 4 \cos 4A \cos 5A \cos 6A$

Let us consider LHS:

$$\cos 3A + \cos 5A + \cos 7A + \cos 15A$$

So now,

$$(\cos 5A + \cos 3A) + (\cos 15A + \cos 7A)$$

By using the formula,

$$\cos A + \cos B = 2 \cos (A+B)/2 \cos (A-B)/2$$

$$(\cos 5A + \cos 3A) + (\cos 15A + \cos 7A)$$

$$= [2 \cos (5A+3A)/2 \cos (5A-3A)/2] + [2 \cos (15A+7A)/2 \cos (15A-7A)/2]$$

$$= [2 \cos 8A/2 \cos 2A/2] + [2 \cos 22A/2 \cos 8A/2]$$

$$= [2 \cos 4A \cos A] + [2 \cos 11A \cos 4A]$$

$$= 2 \cos 4A (\cos 11A + \cos A)$$

Again by using the formula,

$$\cos A + \cos B = 2 \cos (A+B)/2 \cos (A-B)/2$$

$$2 \cos 4A (\cos 11A + \cos A) = 2 \cos 4A [2 \cos (11A+A)/2 \cos (11A-A)/2]$$

$$= 2 \cos 4A [2 \cos 12A/2 \cos 10A/2]$$

$$= 2 \cos 4A [2 \cos 6A \cos 5A]$$

$$= 4 \cos 4A \cos 5A \cos 6A$$

$$= \text{RHS}$$

Hence proved.

(ii) $\cos A + \cos 3A + \cos 5A + \cos 7A = 4 \cos A \cos 2A \cos 4A$

Let us consider LHS:

$$\cos A + \cos 3A + \cos 5A + \cos 7A$$

So now,

$$(\cos 3A + \cos A) + (\cos 7A + \cos 5A)$$

By using the formula,

$$\cos A + \cos B = 2 \cos (A+B)/2 \cos (A-B)/2$$

$$(\cos 3A + \cos A) + (\cos 7A + \cos 5A)$$

$$= [2 \cos (3A+A)/2 \cos (3A-A)/2] + [2 \cos (7A+5A)/2 \cos (7A-5A)/2]$$

$$= [2 \cos 4A/2 \cos 2A/2] + [2 \cos 12A/2 \cos 2A/2]$$

$$= [2 \cos 2A \cos A] + [2 \cos 6A \cos A]$$

$$= 2 \cos A (\cos 6A + \cos 2A)$$

Again by using the formula,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$2 \cos A (\cos 6A + \cos 2A) = 2 \cos A [2 \cos \frac{(6A+2A)}{2} \cos \frac{(6A-2A)}{2}]$$

$$= 2 \cos A [2 \cos 4A \cos 2A]$$

$$= 2 \cos A [2 \cos 4A \cos 2A]$$

$$= 4 \cos A \cos 2A \cos 4A$$

$$= \text{RHS}$$

Hence proved.

$$\text{(iii)} \sin A + \sin 2A + \sin 4A + \sin 5A = 4 \cos \frac{A}{2} \cos \frac{3A}{2} \sin 3A$$

Let us consider LHS:

$$\sin A + \sin 2A + \sin 4A + \sin 5A$$

So now,

$$(\sin 2A + \sin A) + (\sin 5A + \sin 4A)$$

By using the formula,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$(\sin 2A + \sin A) + (\sin 5A + \sin 4A) =$$

$$= [2 \sin \frac{(2A+A)}{2} \cos \frac{(2A-A)}{2}] + [2 \sin \frac{(5A+4A)}{2} \cos \frac{(5A-4A)}{2}]$$

$$= [2 \sin \frac{3A}{2} \cos \frac{A}{2}] + [2 \sin \frac{9A}{2} \cos \frac{A}{2}]$$

$$= 2 \cos \frac{A}{2} (\sin \frac{9A}{2} + \sin \frac{3A}{2})$$

Again by using the formula,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$2 \cos \frac{A}{2} (\sin \frac{9A}{2} + \sin \frac{3A}{2}) = 2 \cos \frac{A}{2} [2 \sin \frac{(\frac{9A}{2} + \frac{3A}{2})}{2} \cos \frac{(\frac{9A}{2} - \frac{3A}{2})}{2}]$$

$$= 2 \cos \frac{A}{2} [2 \sin \frac{(\frac{9A+3A}{2})}{2} \cos \frac{(\frac{9A-3A}{2})}{2}]$$

$$= 2 \cos \frac{A}{2} [2 \sin \frac{12A}{4} \cos \frac{6A}{4}]$$

$$= 2 \cos \frac{A}{2} [2 \sin 3A \cos \frac{3A}{2}]$$

$$= 4 \cos \frac{A}{2} \cos \frac{3A}{2} \sin 3A$$

$$= \text{RHS}$$

Hence proved.

$$\text{(iv)} \sin 3A + \sin 2A - \sin A = 4 \sin A \cos \frac{A}{2} \cos \frac{3A}{2}$$

Let us consider LHS:

$$\sin 3A + \sin 2A - \sin A$$

So now,

$$(\sin 3A - \sin A) + \sin 2A$$

By using the formula,

$$\sin A - \sin B = 2 \cos (A+B)/2 \sin (A-B)/2$$

$$(\sin 3A - \sin A) + \sin 2A = 2 \cos (3A + A)/2 \sin (3A - A)/2 + \sin 2A$$

$$= 2 \cos 4A/2 \sin 2A/2 + \sin 2A$$

We know that, $\sin 2A = 2 \sin A \cos A$

$$= 2 \cos 2A \sin A + 2 \sin A \cos A$$

$$= 2 \sin A (\cos 2A + \cos A)$$

Again by using the formula,

$$\cos A + \cos B = 2 \cos (A+B)/2 \cos (A-B)/2$$

$$2 \sin A (\cos 2A + \cos A) = 2 \sin A [2 \cos (2A+A)/2 \cos (2A-A)/2]$$

$$= 2 \sin A [2 \cos 3A/2 \cos A/2]$$

$$= 4 \sin A \cos A/2 \cos 3A/2$$

$$= \text{RHS}$$

Hence proved.

$$(v) \cos 20^\circ \cos 100^\circ + \cos 100^\circ \cos 140^\circ - \cos 140^\circ \cos 200^\circ = -3/4$$

Let us consider LHS:

$$\cos 20^\circ \cos 100^\circ + \cos 100^\circ \cos 140^\circ - \cos 140^\circ \cos 200^\circ =$$

We shall multiply and divide by 2 we get,

$$= 1/2 [2 \cos 100^\circ \cos 20^\circ + 2 \cos 140^\circ \cos 100^\circ - 2 \cos 200^\circ \cos 140^\circ]$$

We know that $2 \cos A \cos B = \cos (A+B) + \cos (A-B)$

So,

$$= 1/2 [\cos (100^\circ + 20^\circ) + \cos (100^\circ - 20^\circ) + \cos (140^\circ + 100^\circ) + \cos (140^\circ - 100^\circ) - \cos (200^\circ + 140^\circ) - \cos (200^\circ - 140^\circ)]$$

$$= 1/2 [\cos 120^\circ + \cos 80^\circ + \cos 240^\circ + \cos 40^\circ - \cos 340^\circ - \cos 60^\circ]$$

$$= 1/2 [\cos (90^\circ + 30^\circ) + \cos 80^\circ + \cos (180^\circ + 60^\circ) + \cos 40^\circ - \cos (360^\circ - 20^\circ) - \cos 60^\circ]$$

We know, $\cos (180^\circ + A) = -\cos A$, $\cos (90^\circ + A) = -\sin A$, $\cos (360^\circ - A) = \cos A$

So,

$$= 1/2 [-\sin 30^\circ + \cos 80^\circ - \cos 60^\circ + \cos 40^\circ - \cos 20^\circ - \cos 60^\circ]$$

$$= 1/2 [-\sin 30^\circ + \cos 80^\circ + \cos 40^\circ - \cos 20^\circ - 2 \cos 60^\circ]$$

Again by using the formula,

$$\cos A + \cos B = 2 \cos (A+B)/2 \cos (A-B)/2$$

$$= 1/2 [-\sin 30^\circ + 2 \cos (80^\circ + 40^\circ)/2 \cos (80^\circ - 40^\circ)/2 - \cos 20^\circ - 2 \times 1/2]$$

$$\begin{aligned}
&= 1/2 [-\sin 30^\circ + 2 \cos 120^\circ/2 \cos 40^\circ/2 - \cos 20^\circ - 1] \\
&= 1/2 [-\sin 30^\circ + 2 \cos 60^\circ \cos 20^\circ - \cos 20^\circ - 1] \\
&= 1/2 [-1/2 + 2 \times 1/2 \times \cos 20^\circ - \cos 20^\circ - 1] \\
&= 1/2 [-1/2 + \cos 20^\circ - \cos 20^\circ - 1] \\
&= 1/2 [-1/2 - 1] \\
&= 1/2 [-3/2] \\
&= -3/4 \\
&= \text{RHS}
\end{aligned}$$

Hence proved.

(vi) $\sin x/2 \sin 7x/2 + \sin 3x/2 \sin 11x/2 = \sin 2x \sin 5x$

Let us consider LHS:

$$\sin x/2 \sin 7x/2 + \sin 3x/2 \sin 11x/2 =$$

We shall multiply and divide by 2 we get,

$$= 1/2 [2 \sin 7x/2 \sin x/2 + 2 \sin 11x/2 \sin 3x/2]$$

We know that $2 \sin A \sin B = \cos (A-B) - \cos (A+B)$

So,

$$= 1/2 [\cos (7x/2 - x/2) - \cos (7x/2 + x/2) + \cos (11x/2 - 3x/2) - \cos (11x/2 + 3x/2)]$$

$$= 1/2 [\cos (7x-x)/2 - \cos (7x+x)/2 + \cos (11x-3x)/2 - \cos (11x+3x)/2]$$

$$= 1/2 [\cos 6x/2 - \cos 8x/2 + \cos 8x/2 - \cos 14x/2]$$

$$= 1/2 [\cos 3x - \cos 7x]$$

$$= -1/2 [\cos 7x - \cos 3x]$$

Again by using the formula,

$$\cos A - \cos B = -2 \sin (A+B)/2 \sin (A-B)/2$$

$$= -1/2 [-2 \sin (7x+3x)/2 \sin (7x-3x)/2]$$

$$= -1/2 [-2 \sin 10x/2 \sin 4x/2]$$

$$= -1/2 [-2 \sin 5x \sin 2x]$$

$$= -2/-2 \sin 5x \sin 2x$$

$$= \sin 2x \sin 5x$$

$$= \text{RHS}$$

Hence proved.

(vii) $\cos x \cos x/2 - \cos 3x \cos 9x/2 = \sin 4x \sin 7x/2$

Let us consider LHS:

$$\cos x \cos x/2 - \cos 3x \cos 9x/2 =$$

We shall multiply and divide by 2 we get,

$$= 1/2 [2 \cos x \cos x/2 - 2 \cos 9x/2 \cos 3x]$$

We know that $2 \cos A \cos B = \cos (A+B) + \cos (A-B)$

So,

$$= 1/2 [\cos (x + x/2) + \cos (x - x/2) - \cos (9x/2 + 3x) - \cos (9x/2 - 3x)]$$

$$= 1/2 [\cos (2x+x)/2 + \cos (2x-x)/2 - \cos (9x+6x)/2 - \cos (9x-6x)/2]$$

$$= 1/2 [\cos 3x/2 + \cos x/2 - \cos 15x/2 - \cos 3x/2]$$

$$= 1/2 [\cos x/2 - \cos 15x/2]$$

$$= - 1/2 [\cos 15x/2 - \cos x/2]$$

Again by using the formula,

$$\cos A - \cos B = -2 \sin (A+B)/2 \sin (A-B)/2$$

$$= - 1/2 [-2 \sin (15x/2 + x/2)/2 \sin (15x/2 - x/2)/2]$$

$$= -1/2 [-2 \sin (16x/2)/2 \sin (14x/2)/2]$$

$$= -1/2 [-2 \sin 16x/4 \sin 7x/2]$$

$$= - 1/2 [-2 \sin 4x \sin 7x/2]$$

$$= -2/-2 [\sin 4x \sin 7x/2]$$

$$= \sin 4x \sin 7x/2$$

$$= \text{RHS}$$

Hence proved.

7. Prove that:

$$(i) \frac{\sin A + \sin 3A}{\cos A - \cos 3A} = \cot A$$

$$(ii) \frac{\sin 9A - \sin 7A}{\cos 7A - \cos 9A} = \cot 8A$$

$$(iii) \frac{\sin A - \sin B}{\cos A + \cos B} = \tan \frac{A - B}{2}$$

$$(iv) \frac{\sin A + \sin B}{\sin A - \sin B} = \tan\left(\frac{A + B}{2}\right) \cot\left(\frac{A - B}{2}\right)$$

$$(v) \frac{\cos A + \cos B}{\cos B - \cos A} = \cot\left(\frac{A + B}{2}\right) \cot\left(\frac{A - B}{2}\right)$$

Solution:

$$(i) \frac{\sin A + \sin 3A}{\cos A - \cos 3A} = \cot A$$

Solution:

Let us consider LHS:

$$\frac{\sin A + \sin 3A}{\cos A - \cos 3A}$$

By using the formulas,

$$\sin A + \sin B = 2 \sin (A+B)/2 \cos (A-B)/2$$

$$\cos A - \cos B = -2 \sin (A+B)/2 \sin (A-B)/2$$

So now,

$$\begin{aligned} \frac{\sin 3A + \sin A}{\cos A - \cos 3A} &= \frac{2(\sin \frac{A+3A}{2} \cos \frac{3A-A}{2})}{-2(\sin \frac{A+3A}{2} \sin \frac{A-3A}{2})} \\ &= -\frac{\sin \frac{4A}{2} \cos \frac{2A}{2}}{\sin \frac{4A}{2} \sin \frac{-2A}{2}} \\ &= -\frac{\cos A}{-\sin A} \text{ (since, } \sin(-A) = -\sin A) \\ &= \frac{\cos A}{\sin A} \\ &= \cot A \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$(ii) \frac{\sin 9A - \sin 7A}{\cos 7A - \cos 9A} = \cot 8A$$

Solution:

Let us consider LHS:

$$\frac{\sin 9A - \sin 7A}{\cos 7A - \cos 9A}$$

By using the formulas,

$$\sin A - \sin B = 2 \cos \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

$$\cos A - \cos B = -2 \sin \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

So now,

$$\begin{aligned} \frac{\sin 9A - \sin 7A}{\cos 7A - \cos 9A} &= \frac{2(\cos \frac{9A+7A}{2} \sin \frac{9A-7A}{2})}{-2(\sin \frac{7A+9A}{2} \sin \frac{7A-9A}{2})} \\ &= -\frac{(\cos \frac{16A}{2} \sin \frac{2A}{2})}{(\sin \frac{16A}{2} \sin \frac{(-2A)}{2})} \\ &= -\frac{(\cos 8A \sin A)}{(-\sin 8A \sin A)} \text{ (since, } \sin(-A) = -\sin A) \\ &= \frac{\cos 8A}{\sin 8A} \\ &= \cot A \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$(iii) \frac{\sin A - \sin B}{\cos A + \cos B} = \tan \frac{A - B}{2}$$

Solution:

Let us consider LHS:

$$\frac{\sin A - \sin B}{\cos A + \cos B}$$

By using the formulas,

$$\sin A - \sin B = 2 \cos \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

So now,

$$\begin{aligned} \frac{\sin A - \sin B}{\cos A + \cos B} &= \frac{2(\cos \frac{A+B}{2} \sin \frac{A-B}{2})}{2(\cos \frac{A+B}{2} \cos \frac{A-B}{2})} \\ &= \frac{\sin(\frac{A-B}{2})}{\cos(\frac{A-B}{2})} \\ &= \tan \frac{A - B}{2} \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$(iv) \frac{\sin A + \sin B}{\sin A - \sin B} = \tan\left(\frac{A+B}{2}\right) \cot\left(\frac{A-B}{2}\right)$$

Solution:

Let us consider LHS:

$$\frac{\sin A + \sin B}{\sin A - \sin B}$$

By using the formulas,

$$\sin A - \sin B = 2 \cos \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

So now,

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{2\left(\sin \frac{A+B}{2} \cos \frac{A-B}{2}\right)}{2\left(\cos \frac{A+B}{2} \sin \frac{A-B}{2}\right)}$$

$$= \frac{\sin \frac{A+B}{2} \cos \frac{A-B}{2}}{\cos \frac{A+B}{2} \sin \frac{A-B}{2}}$$

$$= \tan\left(\frac{A+B}{2}\right) \cot\left(\frac{A-B}{2}\right)$$

= RHS

Hence proved.

$$(v) \frac{\cos A + \cos B}{\cos B - \cos A} = \cot\left(\frac{A+B}{2}\right) \cot\left(\frac{A-B}{2}\right)$$

Solution:

Let us consider LHS:

$$\frac{\cos A + \cos B}{\cos B - \cos A}$$

By using the formulas,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\cos A - \cos B = -2 \sin \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

So now,

$$\frac{\cos A + \cos B}{\cos B - \cos A} = \frac{2\left(\cos \frac{A+B}{2} \cos \frac{A-B}{2}\right)}{-2\left(\sin \frac{A+B}{2} \sin \frac{B-A}{2}\right)}$$

$$= -\frac{\cos \frac{A+B}{2} \cos \frac{A-B}{2}}{\sin \frac{A+B}{2} \sin\left(-\frac{A-B}{2}\right)}$$

$$\begin{aligned}
&= -\frac{\cos \frac{A+B}{2} \cos \frac{A-B}{2}}{-\sin \frac{A+B}{2} \sin \frac{A-B}{2}} (\text{since, } \sin(-x) = -\sin x) \\
&= \cot\left(\frac{A+B}{2}\right) \cot\left(\frac{A-B}{2}\right) \\
&= \text{RHS}
\end{aligned}$$

Hence proved.

8. Prove that:

- (i) $\frac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \tan 3A$
- (ii) $\frac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A} = \frac{\cos 5A}{\cos 3A}$
- (iii) $\frac{\cos 4A + \cos 3A + \cos 2A}{\sin 4A + \sin 3A + \sin 2A} = \cot 3A$
- (iv) $\frac{\sin 3A + \sin 5A + \sin 7A + \sin 9A}{\cos 3A + \cos 5A + \cos 7A + \cos 9A} = \tan 6A$
- (v) $\frac{\sin 5A - \sin 7A + \sin 8A - \sin 4A}{\cos 4A + \cos 7A - \cos 5A - \cos 8A} = \cot 6A$
- (vi) $\frac{\sin 5A \cos 2A - \sin 6A \cos A}{\sin A \sin 2A - \cos 2A \cos 3A} = \tan A$
- (vii) $\frac{\sin 11A \sin A + \sin 7A \sin 3A}{\cos 11A \sin A + \cos 7A \sin 3A} = \tan 8A$
- (viii) $\frac{\sin 3A \cos 4A - \sin A \cos 2A}{\sin 4A \sin A + \cos 6A \cos A} = \tan 2A$
- (ix) $\frac{\sin A \sin 2A + \sin 3A \sin 6A}{\sin A \cos 2A + \sin 3A \cos 6A} = \tan 5A$
- (x) $\frac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \frac{\sin 3A}{\sin 5A}$

$$(xi) \frac{\sin(\theta + \phi) - 2\sin\theta + \sin(\theta - \phi)}{\cos(\theta + \phi) - 2\cos\theta + \cos(\theta - \phi)} = \tan\theta$$

Solution:

$$(i) \frac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \tan 3A$$

Let us consider LHS:

$$\frac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A}$$

By using the formulas,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

So now,

$$\begin{aligned} \frac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} &= \frac{(\sin 5A + \sin A) + \sin 3A}{(\cos 5A + \cos A) + \cos 3A} \\ &= \frac{\left(2 \sin \frac{5A+A}{2} \cos \frac{5A-A}{2}\right) + \sin 3A}{\left(2 \cos \frac{5A+A}{2} \cos \frac{5A-A}{2}\right) + \cos 3A} \\ &= \frac{\left(2 \sin \frac{6A}{2} \cos \frac{4A}{2}\right) + \sin 3A}{\left(2 \cos \frac{6A}{2} \cos \frac{4A}{2}\right) + \cos 3A} \\ &= \frac{(2 \sin 3A \cos 2A) + \sin 3A}{(2 \cos 3A \cos 2A) + \cos 3A} \\ &= \frac{\sin 3A (2 \cos 2A + 1)}{\cos 3A (2 \cos 2A + 1)} \\ &= \tan 3A \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$(ii) \frac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A} = \frac{\cos 5A}{\cos 3A}$$

Let us consider LHS:

$$\frac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A}$$

By using the formula,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

So now,

$$\begin{aligned} \frac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A} &= \frac{(\cos 7A + \cos 3A) + 2\cos 5A}{(\cos 5A + \cos A) + 2\cos 3A} \\ &= \frac{\left(2\cos \frac{7A+3A}{2} \cos \frac{7A-3A}{2}\right) + 2\cos 5A}{\left(2\cos \frac{5A+A}{2} \cos \frac{5A-A}{2}\right) + 2\cos 3A} \\ &= \frac{\left(2\cos \frac{10A}{2} \cos \frac{4A}{2}\right) + 2\cos 5A}{\left(2\cos \frac{6A}{2} \cos \frac{4A}{2}\right) + 2\cos 3A} \\ &= \frac{(2\cos 5A \cos 2A) + 2\cos 5A}{(2\cos 3A \cos 2A) + 2\cos 3A} \\ &= \frac{2\cos 5A (\cos 2A + 1)}{2\cos 3A (\cos 2A + 1)} \\ &= \frac{\cos 5A}{\cos 3A} \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$(iii) \frac{\cos 4A + \cos 3A + \cos 2A}{\sin 4A + \sin 3A + \sin 2A} = \cot 3A$$

Let us consider LHS:

$$\frac{\cos 4A + \cos 3A + \cos 2A}{\sin 4A + \sin 3A + \sin 2A}$$

By using the formula,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

So now,

$$\begin{aligned} \frac{\cos 4A + \cos 3A + \cos 2A}{\sin 4A + \sin 3A + \sin 2A} &= \frac{(\cos 4A + \cos 2A) + \cos 3A}{(\sin 4A + \sin 2A) + \sin 3A} \\ &= \frac{\left(2 \cos \frac{4A + 2A}{2} \cos \frac{4A - 2A}{2}\right) + \cos 3A}{\left(2 \sin \frac{4A + 2A}{2} \cos \frac{4A - 2A}{2}\right) + \sin 3A} \\ &= \frac{\left(2 \cos \frac{6A}{2} \cos \frac{2A}{2}\right) + \cos 3A}{\left(2 \sin \frac{6A}{2} \cos \frac{2A}{2}\right) + \sin 3A} \\ &= \frac{(2 \cos 3A \cos A) + \cos 3A}{(2 \sin 3A \cos A) + \sin 3A} \\ &= \frac{\cos 3A (2 \cos A + 1)}{\sin 3A (2 \cos A + 1)} \\ &= \cot 3A \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$(iv) \frac{\sin 3A + \sin 5A + \sin 7A + \sin 9A}{\cos 3A + \cos 5A + \cos 7A + \cos 9A} = \tan 6A$$

Let us consider LHS:

$$\frac{\sin 3A + \sin 5A + \sin 7A + \sin 9A}{\cos 3A + \cos 5A + \cos 7A + \cos 9A}$$

By using the formulas,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

So now,

$$\frac{\sin 3A + \sin 5A + \sin 7A + \sin 9A}{\cos 3A + \cos 5A + \cos 7A + \cos 9A}$$

When we rearrange we get,

$$\begin{aligned} &= \frac{(\sin 9A + \sin 3A) + (\sin 7A + \sin 5A)}{(\cos 9A + \cos 3A) + (\cos 7A + \cos 5A)} \\ &= \frac{\left(2 \sin \frac{9A + 3A}{2} \cos \frac{9A - 3A}{2}\right) + \left(2 \sin \frac{7A + 5A}{2} \cos \frac{7A - 5A}{2}\right)}{\left(2 \cos \frac{9A + 3A}{2} \cos \frac{9A - 3A}{2}\right) + \left(2 \cos \frac{7A + 5A}{2} \cos \frac{7A - 5A}{2}\right)} \\ &= \frac{\left(2 \sin \frac{12A}{2} \cos \frac{6A}{2}\right) + \left(2 \sin \frac{12A}{2} \cos \frac{2A}{2}\right)}{\left(2 \cos \frac{12A}{2} \cos \frac{6A}{2}\right) + \left(2 \cos \frac{12A}{2} \cos \frac{2A}{2}\right)} \\ &= \frac{(2 \sin 6A \cos 3A) + (2 \sin 6A \cos A)}{(2 \cos 6A \cos 3A) + (2 \cos 6A \cos A)} \\ &= \frac{2 \sin 6A (\cos 3A + \cos A)}{2 \cos 6A (\cos 3A + \cos A)} \\ &= \frac{\sin 6A}{\cos 6A} \\ &= \tan 6A \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$(v) \frac{\sin 5A - \sin 7A + \sin 8A - \sin 4A}{\cos 4A + \cos 7A - \cos 5A - \cos 8A} = \cot 6A$$

Let us consider LHS:

$$\frac{\sin 5A - \sin 7A + \sin 8A - \sin 4A}{\cos 4A + \cos 7A - \cos 5A - \cos 8A}$$

By using the formulas,

$$\cos A - \cos B = -2 \sin \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

$$\sin A - \sin B = 2 \cos \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

So now,

$$\frac{\sin 5A - \sin 7A + \sin 8A - \sin 4A}{\cos 4A + \cos 7A - \cos 5A - \cos 8A} = \frac{-(\sin 7A - \sin 5A) + (\sin 8A - \sin 4A)}{(\cos 7A - \cos 5A) - (\cos 8A - \cos 4A)}$$

When we rearrange we get,

$$\begin{aligned} &= \frac{-\left(2 \cos \frac{7A + 5A}{2} \sin \frac{7A - 5A}{2}\right) + \left(2 \cos \frac{8A + 4A}{2} \sin \frac{8A - 4A}{2}\right)}{\left(-2 \sin \frac{7A + 5A}{2} \sin \frac{7A - 5A}{2}\right) - \left(-2 \sin \frac{8A + 4A}{2} \sin \frac{8A - 4A}{2}\right)} \\ &= \frac{-\left(2 \cos \frac{12A}{2} \sin \frac{2A}{2}\right) + \left(2 \cos \frac{12A}{2} \sin \frac{4A}{2}\right)}{\left(-2 \sin \frac{12A}{2} \sin \frac{2A}{2}\right) - \left(-2 \sin \frac{12A}{2} \sin \frac{4A}{2}\right)} \\ &= \frac{-(2 \cos 6A \sin A) + (2 \cos 6A \sin 2A)}{-(2 \sin 6A \sin A) + (2 \sin 6A \sin 2A)} \\ &= \frac{2 \cos 6A (-\sin A + \sin 2A)}{2 \sin 6A (-\sin A + \sin 2A)} \\ &= \frac{\cos 6A}{\sin 6A} \end{aligned}$$

$$= \cot 6A$$

$$= \text{RHS}$$

Hence proved.

$$(vi) \frac{\sin 5A \cos 2A - \sin 6A \cos A}{\sin A \sin 2A - \cos 2A \cos 3A} = \tan A$$

Let us consider LHS:

$$\frac{\sin 5A \cos 2A - \sin 6A \cos A}{\sin A \sin 2A - \cos 2A \cos 3A}$$

Let us multiply and divide the above expression by 2 we get,

$$\frac{\sin 5A \cos 2A - \sin 6A \cos A}{\sin A \sin 2A - \cos 2A \cos 3A} = \frac{(2 \sin 5A \cos 2A) - (2 \sin 6A \cos A)}{(2 \sin 2A \sin A) - (2 \cos 3A \cos A)}$$

We know that, $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$,

$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$ and

$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

So now,

$$\begin{aligned} &= \frac{\{\sin(5A + 2A) + \sin(5A - 2A)\} - \{\sin(6A + A) + \sin(6A - A)\}}{\{\cos(2A - A) - \cos(2A + A)\} - \{\cos(3A + 2A) + \cos(3A - 2A)\}} \\ &= \frac{\{\sin 7A + \sin 3A\} - \{\sin 7A + \sin 5A\}}{\{\cos A - \cos 3A\} - \{\cos 5A + \cos A\}} \\ &= \frac{\sin 7A + \sin 3A - \sin 7A - \sin 5A}{\cos A - \cos 3A - \cos 5A - \cos A} \\ &= \frac{\sin 3A - \sin 5A}{-(\cos 5A + \cos 3A)} \\ &= \frac{-(\sin 5A - \sin 3A)}{-(\cos 5A + \cos 3A)} \\ &= \frac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A} \end{aligned}$$

By using the formulas,

$$\sin A - \sin B = 2 \cos \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

So,

$$\begin{aligned} &= \frac{2 \cos \frac{5A + 3A}{2} \sin \frac{5A - 3A}{2}}{2 \cos \frac{5A + 3A}{2} \cos \frac{5A - 3A}{2}} \end{aligned}$$

$$= \frac{2 \cos \frac{5A + 3A}{2} \sin \frac{5A - 3A}{2}}{2 \cos \frac{5A + 3A}{2} \cos \frac{5A - 3A}{2}}$$

$$= \frac{2 \cos \frac{8A}{2} \sin \frac{2A}{2}}{2 \cos \frac{8A}{2} \cos \frac{2A}{2}}$$

$$= \frac{\sin A}{\cos A}$$

$$= \tan A$$

$$= \text{RHS}$$

Hence proved.

$$(vii) \frac{\sin 11A \sin A + \sin 7A \sin 3A}{\cos 11A \sin A + \cos 7A \sin 3A} = \tan 8A$$

Let us consider LHS:

$$\frac{\sin 11A \sin A + \sin 7A \sin 3A}{\cos 11A \sin A + \cos 7A \sin 3A}$$

Let us multiply and divide the above expression by 2 we get,

$$\frac{\sin 11A \sin A + \sin 7A \sin 3A}{\cos 11A \sin A + \cos 7A \sin 3A} = \frac{(2 \sin 11A \sin A) + (2 \sin 7A \sin 3A)}{(2 \cos 11A \sin A) + (2 \cos 7A \sin 3A)}$$

We know that, $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$,

$$2 \cos A \sin B = \sin(A + B) - \sin(A - B)$$

So now,

$$\begin{aligned} &= \frac{\{\cos(11A - A) - \cos(11A + A)\} + \{\cos(7A - 3A) - \cos(7A + 3A)\}}{\{\sin(11A + A) - \sin(11A - A)\} + \{\sin(7A + 3A) - \sin(7A - 3A)\}} \\ &= \frac{\{\cos 10A - \cos 12A\} + \{\cos 4A - \cos 10A\}}{\{\sin 12A - \sin 10A\} + \{\sin 10A - \sin 4A\}} \\ &= \frac{\cos 10A - \cos 12A + \cos 4A - \cos 10A}{\sin 12A - \sin 10A + \sin 10A - \sin 4A} \\ &= \frac{-(\cos 12A - \cos 4A)}{\sin 12A - \sin 4A} \end{aligned}$$

By using the formulas,

$$\sin A - \sin B = 2 \cos \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

$$\cos A - \cos B = -2 \sin \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

so,

$$\begin{aligned} &= -\frac{-2 \sin \frac{12A + 4A}{2} \sin \frac{12A - 4A}{2}}{2 \cos \frac{12A + 4A}{2} \sin \frac{12A - 4A}{2}} \\ &= \frac{\sin \frac{16A}{2} \sin \frac{8A}{2}}{\cos \frac{16A}{2} \sin \frac{8A}{2}} \\ &= \frac{\sin 8A}{\cos 8A} \\ &= \tan 8A \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$\text{(viii)} \frac{\sin 3A \cos 4A - \sin A \cos 2A}{\sin 4A \sin A + \cos 6A \cos A} = \tan 2A$$

Let us consider LHS:

$$\frac{\sin 3A \cos 4A - \sin A \cos 2A}{\sin 4A \sin A + \cos 6A \cos A}$$

Let us multiply and divide the above expression by 2 we get,

$$\frac{\sin 3A \cos 4A - \sin A \cos 2A}{\sin 4A \sin A + \cos 6A \cos A} = \frac{(2 \sin 3A \cos 4A) - (2 \sin A \cos 2A)}{(2 \sin 4A \sin A) + (2 \cos 6A \cos A)}$$

We know that, $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$,

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$$

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

So now,

$$\begin{aligned} &= \frac{\{\sin(3A + 4A) + \sin(3A - 4A)\} - \{\sin(A + 2A) + \sin(A - 2A)\}}{\{\cos(4A - A) - \cos(4A + A)\} + \{\cos(6A + A) + \cos(6A - A)\}} \\ &= \frac{\{\sin 7A + \sin(-A)\} - \{\sin 3A + \sin(-A)\}}{\{\cos 3A - \cos 5A\} + \{\cos 7A + \cos 5A\}} \\ &= \frac{\{\sin 7A - \sin A\} - \{\sin 3A - \sin A\}}{\{\cos 3A - \cos 5A\} + \{\cos 7A + \cos 5A\}} \quad (\text{since, } (-A) = -\sin A) \\ &= \frac{\sin 7A - \sin A - \sin 3A + \sin A}{\cos 3A - \cos 5A + \cos 7A + \cos 5A} \end{aligned}$$

$$= \frac{\sin 7A - \sin 3A}{\cos 7A + \cos 3A}$$

By using the formulas,

$$\sin A - \sin B = 2 \cos \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

so,

$$= \frac{2 \cos \frac{7A + 3A}{2} \sin \frac{7A - 3A}{2}}{2 \cos \frac{7A + 3A}{2} \cos \frac{7A - 3A}{2}}$$

$$= \frac{\cos \frac{10A}{2} \sin \frac{4A}{2}}{\cos \frac{10A}{2} \cos \frac{4A}{2}}$$

$$= \frac{\sin 2A}{\cos 2A}$$

$$= \tan 2A$$

$$= \text{RHS}$$

Hence proved.

$$(ix) \frac{\sin A \sin 2A + \sin 3A \sin 6A}{\sin A \cos 2A + \sin 3A \cos 6A} = \tan 5A$$

Let us consider LHS:

$$\frac{\sin A \sin 2A + \sin 3A \sin 6A}{\sin A \cos 2A + \sin 3A \cos 6A}$$

Let us multiply and divide the above expression by 2 we get,

$$\frac{\sin A \sin 2A + \sin 3A \sin 6A}{\sin A \cos 2A + \sin 3A \cos 6A} = \frac{(2 \sin 2A \sin A) + (2 \sin 6A \sin 3A)}{(2 \sin A \cos 2A) + (2 \sin 3A \cos 6A)}$$

We know that, $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$,

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

So now,

$$\begin{aligned} &= \frac{\{\cos(2A - A) - \cos(2A + A)\} + \{\cos(6A - 3A) - \cos(6A + 3A)\}}{\{\sin(A + 2A) + \sin(A - 2A)\} + \{\sin(3A + 6A) + \sin(3A - 6A)\}} \\ &= \frac{\{\cos A - \cos 3A\} + \{\cos 3A - \cos 9A\}}{\{\sin 3A + \sin(-A)\} + \{\sin 9A + \sin(-3A)\}} \end{aligned}$$

$$\begin{aligned}
&= \frac{\{\cos A - \cos 3A\} + \{\cos 3A - \cos 9A\}}{\{\sin 3A - \sin A\} + \{\sin 9A - \sin 3A\}} \quad (\text{since, } \sin(-A) = -\sin A) \\
&= \frac{\cos A - \cos 3A + \cos 3A - \cos 9A}{\sin 3A - \sin A + \sin 9A - \sin 3A} \\
&= \frac{\cos A - \cos 9A}{\sin 9A - \sin A} \\
&= \frac{-(\cos 9A - \cos A)}{\sin 9A - \sin A}
\end{aligned}$$

By using the formulas,

$$\sin A - \sin B = 2 \cos \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

$$\cos A - \cos B = -2 \sin \frac{(A+B)}{2} \sin \frac{(A-B)}{2}$$

so,

$$\begin{aligned}
&= - \frac{-2 \sin \frac{9A + A}{2} \sin \frac{9A - A}{2}}{2 \cos \frac{9A + A}{2} \sin \frac{9A - A}{2}} \\
&= \frac{\sin \frac{10A}{2} \sin \frac{8A}{2}}{\cos \frac{10A}{2} \sin \frac{8A}{2}} \\
&= \frac{\sin 5A}{\cos 5A} \\
&= \tan 5A
\end{aligned}$$

= RHS

Hence proved.

$$(x) \frac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \frac{\sin 3A}{\sin 5A}$$

Let us consider LHS:

$$\begin{aligned}
&\frac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} \\
&= \frac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \frac{(\sin 5A + \sin A) + 2\sin 3A}{(\sin 7A + \sin 3A) + 2\sin 5A}
\end{aligned}$$

We know that,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

So now,

$$\begin{aligned}
&= \frac{\left(2 \sin \frac{5A + A}{2} \cos \frac{5A - A}{2}\right) + 2 \sin 3A}{\left(2 \sin \frac{7A + 3A}{2} \cos \frac{7A - 3A}{2}\right) + 2 \sin 5A} \\
&= \frac{\left(2 \sin \frac{6A}{2} \cos \frac{4A}{2}\right) + 2 \sin 3A}{\left(2 \sin \frac{10A}{2} \cos \frac{4A}{2}\right) + 2 \sin 5A} \\
&= \frac{(2 \sin 3A \cos 2A) + 2 \sin 3A}{(2 \sin 5A \cos 2A) + 2 \sin 5A} \\
&= \frac{2 \sin 3A (\cos 2A + 1)}{2 \sin 5A (\cos 2A + 1)} \\
&= \frac{\sin 3A}{\sin 5A} \\
&= \text{RHS}
\end{aligned}$$

Hence proved.

$$(xi) \frac{\sin(\theta + \phi) - 2\sin\theta + \sin(\theta - \phi)}{\cos(\theta + \phi) - 2\cos\theta + \cos(\theta - \phi)} = \tan\theta$$

Let us consider LHS:

$$\begin{aligned}
&\frac{\sin(\theta + \phi) - 2\sin\theta + \sin(\theta - \phi)}{\cos(\theta + \phi) - 2\cos\theta + \cos(\theta - \phi)} \\
&= \frac{\sin(\theta + \phi) - 2\sin\theta + \sin(\theta - \phi)}{\cos(\theta + \phi) - 2\cos\theta + \cos(\theta - \phi)} = \frac{\{\sin(\theta + \phi) + \sin(\theta - \phi)\} - 2\sin\theta}{\{\cos(\theta + \phi) + \cos(\theta - \phi)\} - 2\cos\theta}
\end{aligned}$$

We know that,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

So now,

$$\begin{aligned}
&= \frac{\left(2 \sin \frac{\theta + \phi + \theta - \phi}{2} \cos \frac{\theta + \phi - \theta + \phi}{2}\right) - 2 \sin \theta}{\left(2 \cos \frac{\theta + \phi + \theta - \phi}{2} \cos \frac{\theta + \phi - \theta + \phi}{2}\right) - 2 \cos \theta} \\
&= \frac{\left(2 \sin \frac{2\theta}{2} \cos \frac{2\phi}{2}\right) - 2 \sin \theta}{\left(2 \cos \frac{2\theta}{2} \cos \frac{2\phi}{2}\right) - 2 \cos \theta}
\end{aligned}$$

$$\begin{aligned}
&= \frac{(2 \sin \theta \cos \Phi) - 2 \sin \theta}{(2 \cos \theta \cos \Phi) - 2 \cos \theta} \\
&= \frac{2 \sin \theta (\cos \Phi - 1)}{2 \cos \theta (\cos \Phi - 1)} \\
&= \frac{\sin \theta}{\cos \theta} \\
&= \tan \theta \\
&= \text{RHS}
\end{aligned}$$

Hence proved.

9. Prove that:

(i) $\sin \alpha + \sin \beta + \sin \gamma - \sin (\alpha + \beta + \gamma) = 4 \sin (\alpha + \beta)/2 \sin (\beta + \gamma)/2 \sin (\alpha + \gamma)/2$

(ii) $\cos (A + B + C) + \cos (A - B + C) + \cos (A + B - C) + \cos (-A + B + C) = 4 \cos A \cos B \cos C$

Solution:

(i) $\sin \alpha + \sin \beta + \sin \gamma - \sin (\alpha + \beta + \gamma) = 4 \sin (\alpha + \beta)/2 \sin (\beta + \gamma)/2 \sin (\alpha + \gamma)/2$

Let us consider LHS:

$$\sin \alpha + \sin \beta + \sin \gamma - \sin (\alpha + \beta + \gamma)$$

By using the formulas,

$$\sin A + \sin B = 2 \sin (A+B)/2 \cos (A-B)/2$$

$$\sin A - \sin B = 2 \cos (A+B)/2 \sin (A-B)/2$$

$$\begin{aligned}
&\sin \alpha + \sin \beta + \sin \gamma - \sin (\alpha + \beta + \gamma) \\
&= (\sin \alpha + \sin \beta) + \{\sin \gamma - \sin (\alpha + \beta + \gamma)\} \\
&= \left(2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}\right) + \left(2 \cos \frac{\gamma + \alpha + \beta + \gamma}{2} \sin \frac{\gamma - \alpha - \beta - \gamma}{2}\right) \\
&= \left(2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}\right) + \left(2 \cos \frac{\alpha + \beta + 2\gamma}{2} \sin \frac{-(\alpha + \beta)}{2}\right) \\
&= \left(2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}\right) - \left(2 \cos \frac{\alpha + \beta + 2\gamma}{2} \sin \frac{\alpha + \beta}{2}\right) \quad (\text{since, } \sin (-A) = -\sin A) \\
&= 2 \sin \frac{\alpha + \beta}{2} \left(\cos \frac{\alpha - \beta}{2} - \cos \frac{\alpha + \beta + 2\gamma}{2}\right)
\end{aligned}$$

Again by using the formula,

$$\cos A - \cos B = -2 \sin (A+B)/2 \sin (A-B)/2$$

$$\begin{aligned}
&= 2 \sin \frac{\alpha + \beta}{2} \left(-2 \sin \frac{\frac{\alpha - \beta}{2} + \frac{\alpha + \beta + 2\gamma}{2}}{2} \sin \frac{\frac{\alpha - \beta}{2} - \frac{\alpha + \beta + 2\gamma}{2}}{2} \right) \\
&= 2 \sin \frac{\alpha + \beta}{2} \left(-2 \sin \frac{\frac{\alpha - \beta + \alpha + \beta + 2\gamma}{2}}{2} \sin \frac{\frac{\alpha - \beta - (\alpha + \beta + 2\gamma)}{2}}{2} \right) \\
&= 2 \sin \frac{\alpha + \beta}{2} \left(-2 \sin \frac{\frac{2\alpha + 2\gamma}{2}}{2} \sin \frac{\frac{\alpha - \beta - \alpha - \beta - 2\gamma}{2}}{2} \right) \\
&= 2 \sin \frac{\alpha + \beta}{2} \left(-2 \sin \frac{\frac{2\alpha + 2\gamma}{2}}{2} \sin \frac{\frac{-2\beta - 2\gamma}{2}}{2} \right) \\
&= 2 \sin \frac{\alpha + \beta}{2} \left(-2 \sin \frac{\frac{2(\alpha + \gamma)}{2}}{2} \sin \frac{\frac{-2(\beta + \gamma)}{2}}{2} \right) \\
&= 2 \sin \frac{\alpha + \beta}{2} \left(-2 \sin \frac{\alpha + \gamma}{2} \sin \frac{-(\beta + \gamma)}{2} \right) \\
&= 2 \sin \frac{\alpha + \beta}{2} \left(2 \sin \frac{\alpha + \gamma}{2} \sin \frac{\beta + \gamma}{2} \right) \\
&= 4 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta + \gamma}{2} \sin \frac{\alpha + \gamma}{2}
\end{aligned}$$

= RHS

Hence proved.

(ii) $\cos(A + B + C) + \cos(A - B + C) + \cos(A + B - C) + \cos(-A + B + C) = 4 \cos A \cos B \cos C$

Let us consider LHS:

$$\cos(A + B + C) + \cos(A - B + C) + \cos(A + B - C) + \cos(-A + B + C)$$

so,

$$\begin{aligned}
&\cos(A + B + C) + \cos(A - B + C) + \cos(A + B - C) + \cos(-A + B + C) = \\
&= \{\cos(A + B + C) + \cos(A - B + C)\} + \{\cos(A + B - C) + \cos(-A + B + C)\}
\end{aligned}$$

By using the formula,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$= \left\{ 2 \cos \frac{(A + B + C) + (A - B + C)}{2} \cos \frac{(A + B + C) - (A - B + C)}{2} \right\} \\ + \left\{ 2 \cos \frac{(A + B - C) + (-A + B + C)}{2} \cos \frac{(A + B - C) - (-A + B + C)}{2} \right\}$$

$$= \left\{ 2 \cos \frac{A + B + C + A - B + C}{2} \cos \frac{A + B + C - A + B - C}{2} \right\} \\ + \left\{ 2 \cos \frac{A + B - C - A + B + C}{2} \cos \frac{A + B - C + A - B - C}{2} \right\}$$

$$= \left\{ 2 \cos \frac{2A + 2C}{2} \cos \frac{2B}{2} \right\} + \left\{ 2 \cos \frac{2B}{2} \cos \frac{2A - 2C}{2} \right\}$$

$$= \left\{ 2 \cos \frac{2(A + C)}{2} \cos \frac{2B}{2} \right\} + \left\{ 2 \cos \frac{2B}{2} \cos \frac{2(A - C)}{2} \right\}$$

$$= 2 \cos (A + C) \cos B + 2 \cos B \cos (A - C)$$

$$= 2 \cos B \{ \cos (A + C) + \cos (A - C) \}$$

By using the formula,

$$\cos A + \cos B = 2 \cos \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

$$= 2 \cos B \left\{ 2 \cos \frac{A + C + (A - C)}{2} \cos \frac{A + C - (A - C)}{2} \right\}$$

$$= 2 \cos B \left\{ 2 \cos \frac{A + C + A - C}{2} \cos \frac{A + C - A + C}{2} \right\}$$

$$= 2 \cos B \left\{ 2 \cos \frac{2A}{2} \cos \frac{2C}{2} \right\}$$

$$= 4 \cos A \cos B \cos C$$

$$= \text{RHS}$$

Hence proved.