

RD Sharma Solutions for Class 11 Maths Chapter 27 Hyperbola

1. The equation of the directrix of a hyperbola is $x - y + 3 = 0$. Its focus is $(-1, 1)$ and eccentricity 3. Find the equation of the hyperbola.

Solution:

Given:

The equation of the directrix of a hyperbola $\Rightarrow x - y + 3 = 0$.

Focus = $(-1, 1)$ and

Eccentricity = 3

Now, let us find the equation of the hyperbola

Let 'M' be the point on directrix and P(x, y) be any point of the hyperbola.

By using the formula,

$$e = PF/PM$$

PF = ePM [where, e is eccentricity, PM is perpendicular from any point P on hyperbola to the directrix]

So,

$$\sqrt{(x+1)^2 + (y-1)^2} = 3 \left| \frac{(x-y+3)}{\sqrt{1^2 + (-1)^2}} \right|$$

$$\sqrt{(x+1)^2 + (y-1)^2} = 3 \left| \frac{(x-y+3)}{\sqrt{1+1}} \right|$$

By squaring on both sides we get

$$\left(\sqrt{(x+1)^2 + (y-1)^2} \right)^2 = \left(3 \left| \frac{(x-y+3)}{\sqrt{1+1}} \right| \right)^2$$

$$(x+1)^2 + (y-1)^2 = \frac{3^2(x-y+3)^2}{2}$$

[We know that $(a-b)^2 = a^2 + b^2 + 2ab$ & $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$]

So, $2\{x^2 + 1 + 2x + y^2 + 1 - 2y\} = 9\{x^2 + y^2 + 9 + 6x - 6y - 2xy\}$

$$2x^2 + 2 + 4x + 2y^2 + 2 - 4y = 9x^2 + 9y^2 + 81 + 54x - 54y - 18xy$$

$$2x^2 + 4 + 4x + 2y^2 - 4y - 9x^2 - 9y^2 - 81 - 54x + 54y + 18xy = 0$$

$$-7x^2 - 7y^2 - 50x + 50y + 18xy - 77 = 0$$

$$7(x^2 + y^2) - 18xy + 50x - 50y + 77 = 0$$

$\therefore$ The equation of hyperbola is $7(x^2 + y^2) - 18xy + 50x - 50y + 77 = 0$

2. Find the equation of the hyperbola whose

- (i) focus is (0, 3), directrix is $x + y - 1 = 0$ and eccentricity = 2
(ii) focus is (1, 1), directrix is $3x + 4y + 8 = 0$ and eccentricity = 2
(iii) focus is (1, 1) directrix is $2x + y = 1$ and eccentricity = $\sqrt{3}$
(iv) focus is (2, -1), directrix is $2x + 3y = 1$ and eccentricity = 2
(v) focus is (a, 0), directrix is $2x + 3y = 1$ and eccentricity = 2
(vi) focus is (2, 2), directrix is $x + y = 9$ and eccentricity = 2

Solution:

- (i) focus is (0, 3), directrix is $x + y - 1 = 0$ and eccentricity = 2

Given:

Focus = (0, 3)

Directrix $\Rightarrow x + y - 1 = 0$

Eccentricity = 2

Now, let us find the equation of the hyperbola

Let 'M' be the point on directrix and P(x, y) be any point of the hyperbola.

By using the formula,

$$e = PF/PM$$

PF = ePM [where, e is eccentricity, PM is perpendicular from any point P on hyperbola to the directrix]

So,

$$\sqrt{(x-0)^2 + (y-3)^2} = 2 \left| \frac{(x+y-1)}{\sqrt{1^2+1^2}} \right|$$

$$\sqrt{(x-0)^2 + (y-3)^2} = 2 \left| \frac{(x+y-1)}{\sqrt{1+1}} \right|$$

By squaring on both sides we get

$$\left(\sqrt{(x-0)^2 + (y-3)^2} \right)^2 = \left(2 \left| \frac{(x+y-1)}{\sqrt{1+1}} \right| \right)^2$$

$$(x-0)^2 + (y-3)^2 = \frac{2^2(x+y-1)^2}{2}$$

[We know that $(a-b)^2 = a^2 + b^2 - 2ab$ & $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$]

So, $2\{x^2 + y^2 + 9 - 6y\} = 4\{x^2 + y^2 + 1 - 2x - 2y + 2xy\}$

$$2x^2 + 2y^2 + 18 - 12y = 4x^2 + 4y^2 + 4 - 8x - 8y + 8xy$$

$$2x^2 + 2y^2 + 18 - 12y - 4x^2 - 4y^2 - 4 - 8x + 8y - 8xy = 0$$

$$-2x^2 - 2y^2 - 8x - 4y - 8xy + 14 = 0$$

$$-2(x^2 + y^2 - 4x + 2y + 4xy - 7) = 0$$

$$x^2 + y^2 - 4x + 2y + 4xy - 7 = 0$$

∴ The equation of hyperbola is $x^2 + y^2 - 4x + 2y + 4xy - 7 = 0$

(ii) focus is (1, 1), directrix is $3x + 4y + 8 = 0$ and eccentricity = 2

Focus = (1, 1)

Directrix $\Rightarrow 3x + 4y + 8 = 0$

Eccentricity = 2

Now, let us find the equation of the hyperbola

Let 'M' be the point on directrix and P(x, y) be any point of the hyperbola.

By using the formula,

$$e = PF/PM$$

PF = ePM [where, e is eccentricity, PM is perpendicular from any point P on hyperbola to the directrix]

So,

$$\sqrt{(x-0)^2 + (y-3)^2} = 2 \left| \frac{(x+y-1)}{\sqrt{1^2 + 1^2}} \right|$$

$$\sqrt{(x-0)^2 + (y-3)^2} = 2 \left| \frac{(x+y-1)}{\sqrt{1+1}} \right|$$

By squaring on both sides we get

$$\left(\sqrt{(x-0)^2 + (y-3)^2} \right)^2 = \left(2 \left| \frac{(x+y-1)}{\sqrt{1+1}} \right| \right)^2$$

$$(x-0)^2 + (y-3)^2 = \frac{2^2(x+y-1)^2}{2}$$

[We know that $(a-b)^2 = a^2 + b^2 - 2ab$ & $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$]

$$25\{x^2 + 1 - 2x + y^2 + 1 - 2y\} = 4\{9x^2 + 16y^2 + 64 + 24xy + 64y + 48x\}$$

$$25x^2 + 25 - 50x + 25y^2 + 25 - 50y = 36x^2 + 64y^2 + 256 + 96xy + 256y + 192x$$

$$25x^2 + 25 - 50x + 25y^2 + 25 - 50y - 36x^2 - 64y^2 - 256 - 96xy - 256y - 192x = 0$$

$$-11x^2 - 39y^2 - 242x - 306y - 96xy - 206 = 0$$

$$11x^2 + 96xy + 39y^2 + 242x + 306y + 206 = 0$$

∴ The equation of hyperbola is $11x^2 + 96xy + 39y^2 + 242x + 306y + 206 = 0$

(iii) focus is (1, 1) directrix is $2x + y = 1$ and eccentricity = $\sqrt{3}$

Given:

Focus = (1, 1)

Directrix $\Rightarrow 2x + y = 1$

Eccentricity $= \sqrt{3}$

Now, let us find the equation of the hyperbola

Let 'M' be the point on directrix and P(x, y) be any point of the hyperbola.

By using the formula,

$$e = PF/PM$$

PF = ePM [where, e is eccentricity, PM is perpendicular from any point P on hyperbola to the directrix]

So,

$$\sqrt{(x-1)^2 + (y-1)^2} = \sqrt{3} \left| \frac{(2x+y-1)}{\sqrt{2^2+1^2}} \right|$$

$$\sqrt{(x-1)^2 + (y-1)^2} = \sqrt{3} \left| \frac{(2x+y-1)}{\sqrt{4+1}} \right|$$

By squaring on both sides we get

$$\left(\sqrt{(x-1)^2 + (y-1)^2} \right)^2 = \left(\sqrt{3} \left| \frac{(2x+y-1)}{\sqrt{5}} \right| \right)^2$$

$$(x-1)^2 + (y-1)^2 = \frac{3(2x+y-1)^2}{5}$$

[We know that $(a-b)^2 = a^2 + b^2 + 2ab$ & $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$]

$$5\{x^2 + 1 - 2x + y^2 + 1 - 2y\} = 3\{4x^2 + y^2 + 1 + 4xy - 2y - 4x\}$$

$$5x^2 + 5 - 10x + 5y^2 + 5 - 10y = 12x^2 + 3y^2 + 3 + 12xy - 6y - 12x$$

$$5x^2 + 5 - 10x + 5y^2 + 5 - 10y - 12x^2 - 3y^2 - 3 - 12xy + 6y + 12x = 0$$

$$-7x^2 + 2y^2 + 2x - 4y - 12xy + 7 = 0$$

$$7x^2 + 12xy - 2y^2 - 2x + 4y - 7 = 0$$

$\therefore$ The equation of hyperbola is $7x^2 + 12xy - 2y^2 - 2x + 4y - 7 = 0$

(iv) focus is (2, -1), directrix is $2x + 3y = 1$ and eccentricity = 2

Given:

$$\text{Focus} = (2, -1)$$

$$\text{Directrix} \Rightarrow 2x + 3y = 1$$

$$\text{Eccentricity} = 2$$

Now, let us find the equation of the hyperbola

Let 'M' be the point on directrix and P(x, y) be any point of the hyperbola.

By using the formula,

$$e = PF/PM$$

PF = ePM [where, e is eccentricity, PM is perpendicular from any point P on hyperbola to the directrix]

So,

$$\sqrt{(x-2)^2 + (y+1)^2} = 2 \left| \frac{(2x+3y-1)}{\sqrt{2^2+3^2}} \right|$$

$$\sqrt{(x-2)^2 + (y+1)^2} = 2 \left| \frac{(2x+3y-1)}{\sqrt{4+9}} \right|$$

By squaring on both sides we get

$$\left(\sqrt{(x-2)^2 + (y+1)^2} \right)^2 = \left(2 \left| \frac{(2x+3y-1)}{\sqrt{13}} \right| \right)^2$$

$$(x-2)^2 + (y+1)^2 = \frac{4(2x+3y-1)^2}{13}$$

$$(x-2)^2 + (y+1)^2 = \frac{4(2x+3y-1)^2}{13}$$

[We know that $(a-b)^2 = a^2 + b^2 - 2ab$ & $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$]

$$13\{x^2 + 4 - 4x + y^2 + 1 + 2y\} = 4\{4x^2 + 9y^2 + 1 + 12xy - 6y - 4x\}$$

$$13x^2 + 52 - 52x + 13y^2 + 13 + 26y = 16x^2 + 36y^2 + 4 + 48xy - 24y - 16x$$

$$13x^2 + 52 - 52x + 13y^2 + 13 + 26y - 16x^2 - 36y^2 - 4 - 48xy + 24y + 16x = 0$$

$$-3x^2 - 23y^2 - 36x + 50y - 48xy + 61 = 0$$

$$3x^2 + 23y^2 + 48xy + 36x - 50y - 61 = 0$$

$$\therefore \text{The equation of hyperbola is } 3x^2 + 23y^2 + 48xy + 36x - 50y - 61 = 0$$

(v) focus is (a, 0), directrix is $2x + 3y = 1$ and eccentricity = 2

Given:

$$\text{Focus} = (a, 0)$$

$$\text{Directrix} \Rightarrow 2x + 3y = 1$$

$$\text{Eccentricity} = 2$$

Now, let us find the equation of the hyperbola

Let 'M' be the point on directrix and P(x, y) be any point of the hyperbola.

By using the formula,

$$e = PF/PM$$

PF = ePM [where, e is eccentricity, PM is perpendicular from any point P on hyperbola to the directrix]

So,

$$\sqrt{(x-a)^2 + (y-0)^2} = \frac{4}{3} \left| \frac{(2x-y+a)}{\sqrt{2^2 + (-1)^2}} \right|$$

$$\sqrt{(x-a)^2 + (y)^2} = \frac{4}{3} \left| \frac{(2x-y+a)}{\sqrt{4+1}} \right|$$

By squaring on both sides we get

$$\left(\sqrt{(x-a)^2 + (y)^2} \right)^2 = \left(\frac{4}{3} \left| \frac{(2x-y+a)}{\sqrt{5}} \right| \right)^2$$

$$(x-a)^2 + (y)^2 = \frac{16(2x-y+a)^2}{9 \times 5}$$

[We know that $(a-b)^2 = a^2 + b^2 + 2ab$ & $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$]

$$45\{x^2 + a^2 - 2ax + y^2\} = 16\{4x^2 + y^2 + a^2 - 4xy - 2ay + 4ax\}$$

$$45x^2 + 45a^2 - 90ax + 45y^2 = 64x^2 + 16y^2 + 16a^2 - 64xy - 32ay + 64ax$$

$$45x^2 + 45a^2 - 90ax + 45y^2 - 64x^2 - 16y^2 - 16a^2 + 64xy + 32ay - 64ax = 0$$

$$19x^2 - 29y^2 + 154ax - 32ay - 64xy - 29a^2 = 0$$

$$\therefore \text{The equation of hyperbola is } 19x^2 - 29y^2 + 154ax - 32ay - 64xy - 29a^2 = 0$$

(vi) focus is (2, 2), directrix is $x + y = 9$ and eccentricity = 2

Given:

Focus = (2, 2)

Directrix $\Rightarrow x + y = 9$

Eccentricity = 2

Now, let us find the equation of the hyperbola

Let 'M' be the point on directrix and P(x, y) be any point of the hyperbola.

By using the formula,

$$e = PF/PM$$

PF = ePM [where, e is eccentricity, PM is perpendicular from any point P on hyperbola to the directrix]

So,

$$\sqrt{(x-2)^2 + (y-2)^2} = 2 \left| \frac{(x+y-9)}{\sqrt{1^2 + 1^2}} \right|$$

$$\sqrt{(x-2)^2 + (y-2)^2} = 2 \left| \frac{(x+y-9)}{\sqrt{1+1}} \right|$$

By squaring on both sides we get

$$\left(\sqrt{(x-2)^2 + (y-2)^2} \right)^2 = \left(2 \left| \frac{(x+y-9)}{\sqrt{2}} \right| \right)^2$$

$$(x-2)^2 + (y-2)^2 = \frac{4(x+y-9)^2}{2}$$

[We know that $(a-b)^2 = a^2 + b^2 + 2ab$ & $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$]

$$x^2 + 4 - 4x + y^2 + 4 - 4y = 2\{x^2 + y^2 + 81 + 2xy - 18y - 18x\}$$

$$x^2 - 4x + y^2 + 8 - 4y = 2x^2 + 2y^2 + 162 + 4xy - 36y - 36x$$

$$x^2 - 4x + y^2 + 8 - 4y - 2x^2 - 2y^2 - 162 - 4xy + 36y + 36x = 0$$

$$-x^2 - y^2 + 32x + 32y + 4xy - 154 = 0$$

$$x^2 + 4xy + y^2 - 32x - 32y + 154 = 0$$

∴ The equation of hyperbola is $x^2 + 4xy + y^2 - 32x - 32y + 154 = 0$

3. Find the eccentricity, coordinates of the foci, equations of directrices and length of the latus-rectum of the hyperbola.

(i) $9x^2 - 16y^2 = 144$

(ii) $16x^2 - 9y^2 = -144$

(iii) $4x^2 - 3y^2 = 36$

(iv) $3x^2 - y^2 = 4$

(v) $2x^2 - 3y^2 = 5$

Solution:

(i) $9x^2 - 16y^2 = 144$

Given:

The equation $\Rightarrow 9x^2 - 16y^2 = 144$

The equation can be expressed as:

$$\frac{9x^2}{144} - \frac{16y^2}{144} = 1$$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

$$\frac{x^2}{4^2} - \frac{y^2}{3^2} = 1$$

The obtained equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Where, $a^2 = 16$, $b^2 = 9$ i.e., $a = 4$ and $b = 3$

Eccentricity is given by:

$$\begin{aligned} e &= \sqrt{1 + \frac{a^2}{b^2}} \\ &= \sqrt{1 + \frac{9}{16}} \\ &= \sqrt{\frac{25}{16}} \\ &= \frac{5}{4} \end{aligned}$$

Foci: The coordinates of the foci are $(0, \pm be)$

$$(0, \pm be) = (0, \pm 4(5/4))$$

$$= (0, \pm 5)$$

The equation of directrices is given as:

$$\begin{aligned} x &= \pm \frac{a}{e} \\ \Rightarrow x &= \pm \frac{4}{5/4} \\ \Rightarrow x &= \pm \frac{16}{5} \\ \Rightarrow 5x &= \pm 16 \end{aligned}$$

$$5x \mp 16 = 0$$

The length of latus-rectum is given as:

$$2b^2/a$$

$$= 2(9)/4$$

$$= 9/2$$

$$(ii) 16x^2 - 9y^2 = -144$$

Given:

$$\text{The equation } \Rightarrow 16x^2 - 9y^2 = -144$$

The equation can be expressed as:

$$\frac{9y^2}{144} - \frac{16x^2}{144} = 1$$

$$\frac{y^2}{16} - \frac{x^2}{9} = 1$$

$$\frac{x^2}{3^2} - \frac{y^2}{4^2} = -1$$

The obtained equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Where, $a^2 = 9$, $b^2 = 16$ i.e., $a = 3$ and $b = 4$

Eccentricity is given by:

$$\begin{aligned} e &= \sqrt{1 + \frac{a^2}{b^2}} \\ &= \sqrt{1 + \frac{9}{16}} \\ &= \sqrt{\frac{25}{16}} \\ &= \frac{5}{4} \end{aligned}$$

Foci: The coordinates of the foci are $(0, \pm be)$

$$(0, \pm be) = (0, \pm 4(5/4))$$

$$= (0, \pm 5)$$

The equation of directrices is given as:

$$y = \pm \frac{b}{e}$$

$$\Rightarrow y = \pm \frac{4}{\frac{4}{5}}$$

$$\Rightarrow y = \pm \frac{16}{5}$$

$$\Rightarrow 5y = \pm 16$$

$$5y \mp 16 = 0$$

The length of latus-rectum is given as:

$$2a^2/b$$

$$= 2(9)/4$$

$$= 9/2$$

(iii) $4x^2 - 3y^2 = 36$

Given:

The equation $\Rightarrow 4x^2 - 3y^2 = 36$

The equation can be expressed as:

$$\frac{4x^2}{36} - \frac{3y^2}{36} = 1$$

$$\frac{x^2}{9} - \frac{y^2}{12} = 1$$

$$\frac{x^2}{3^2} - \frac{y^2}{(\sqrt{12})^2} = 1$$

The obtained equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Where, $a^2 = 9$, $b^2 = 12$ i.e., $a = 3$ and $b = \sqrt{12}$

Eccentricity is given by:

$$e = \sqrt{1 + \frac{b^2}{a^2}}$$

$$= \sqrt{1 + \frac{12}{9}}$$

$$= \sqrt{\frac{21}{9}}$$

$$= \sqrt{\frac{7}{3}}$$

Foci: The coordinates of the foci are $(\pm ae, 0)$

$$\begin{aligned}\pm ae &= \pm 3 \times \sqrt{\frac{7}{3}} \\ &= \pm 3 \times \frac{\sqrt{7}}{\sqrt{3}} \\ &= \pm \sqrt{3} \times \sqrt{7} \\ &= \pm \sqrt{21}\end{aligned}$$

$$(\pm ae, 0) = (\pm \sqrt{21}, 0)$$

The equation of directrices is given as:

$$\begin{aligned}x &= \frac{\pm a}{e} \\ x &= \pm 3 \times \frac{1}{\frac{\sqrt{7}}{\sqrt{3}}} \\ &= \pm \frac{3\sqrt{3}}{\sqrt{7}} \\ \sqrt{7}x \mp 3\sqrt{3} &= 0\end{aligned}$$

The length of latus-rectum is given as:

$$\begin{aligned}2b^2/a \\ &= 2(12)/3 \\ &= 24/3 \\ &= 8\end{aligned}$$

$$\text{(iv) } 3x^2 - y^2 = 4$$

Given:

$$\text{The equation } \Rightarrow 3x^2 - y^2 = 4$$

The equation can be expressed as:

$$\frac{3x^2}{4} - \frac{y^2}{4} = 1$$

$$\frac{x^2}{\frac{4}{3}} - \frac{y^2}{4} = 1$$

$$\frac{x^2}{\left(\frac{2}{\sqrt{3}}\right)^2} - \frac{y^2}{(2)^2} = 1$$

The obtained equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Where, $a = 2/\sqrt{3}$ and $b = 2$

Eccentricity is given by:

$$\begin{aligned} e &= \sqrt{1 + \frac{b^2}{a^2}} \\ &= \sqrt{1 + \frac{4}{\frac{4}{3}}} \\ &= \sqrt{1 + 3} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

Foci: The coordinates of the foci are $(\pm ae, 0)$

$$(\pm ae, 0) = \pm(2/\sqrt{3})(2) = \pm 4/\sqrt{3}$$

$$(\pm ae, 0) = (\pm 4/\sqrt{3}, 0)$$

The equation of directrices is given as:

$$\begin{aligned} x &= \pm \frac{a}{e} \\ \Rightarrow x &= \pm \frac{\frac{2}{\sqrt{3}}}{2} \\ \Rightarrow x &= \pm \frac{1}{\sqrt{3}} \\ \Rightarrow \sqrt{3}x &= \pm 1 \\ \sqrt{3}x \mp 1 &= 0 \end{aligned}$$

The length of latus-rectum is given as:

$$2b^2/a$$

$$= 2(4)/[2/\sqrt{3}]$$

$$= 4\sqrt{3}$$

$$(v) 2x^2 - 3y^2 = 5$$

Given:

$$\text{The equation } \Rightarrow 2x^2 - 3y^2 = 5$$

The equation can be expressed as:

$$\begin{aligned} \frac{2x^2}{5} - \frac{3y^2}{5} &= 1 \\ \frac{x^2}{\frac{5}{2}} - \frac{y^2}{\frac{5}{3}} &= 1 \\ \frac{x^2}{\left(\frac{\sqrt{5}}{\sqrt{2}}\right)^2} - \frac{y^2}{\left(\frac{\sqrt{5}}{\sqrt{3}}\right)^2} &= 1 \end{aligned}$$

The obtained equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Where, $a = \sqrt{5}/\sqrt{2}$ and $b = \sqrt{5}/\sqrt{3}$

Eccentricity is given by:

$$\begin{aligned} e &= \sqrt{1 + \frac{b^2}{a^2}} \\ &= \sqrt{1 + \frac{\frac{5}{3}}{\frac{5}{2}}} \\ &= \sqrt{1 + \frac{5}{3} \times \frac{2}{5}} \\ &= \sqrt{1 + \frac{2}{3}} \\ &= \sqrt{\frac{5}{3}} \end{aligned}$$

Foci: The coordinates of the foci are $(\pm ae, 0)$

$$\begin{aligned} \pm ae &= \pm \sqrt{\frac{5}{2}} \times \sqrt{\frac{5}{3}} \\ &= \pm \frac{5}{\sqrt{6}} \end{aligned}$$

$$(\pm ae, 0) = (\pm 5/\sqrt{6}, 0)$$

The equation of directrices is given as:

$$\begin{aligned} x &= \pm \frac{a}{e} \\ \Rightarrow x &= \pm \frac{\frac{\sqrt{5}}{\sqrt{2}}}{\frac{\sqrt{5}}{\sqrt{3}}} \\ \Rightarrow x &= \pm \frac{1}{\sqrt{6}} \\ \Rightarrow \sqrt{6}x &= \pm 1 \\ \sqrt{6}x \mp 1 &= 0 \end{aligned}$$

The length of latus-rectum is given as:

$$2b^2/a$$

$$\begin{aligned} &= \frac{2 \left(\frac{\sqrt{5}}{\sqrt{3}} \right)^2}{\frac{\sqrt{5}}{\sqrt{2}}} \\ &= \frac{2 \times \frac{5}{3}}{\frac{\sqrt{5}}{\sqrt{2}}} \\ &= \frac{2\sqrt{10}}{3} \end{aligned}$$

4. Find the axes, eccentricity, latus-rectum and the coordinates of the foci of the hyperbola $25x^2 - 36y^2 = 225$.

Solution:

Given:

$$\text{The equation} \Rightarrow 25x^2 - 36y^2 = 225$$

The equation can be expressed as:

$$\frac{25x^2}{225} - \frac{36y^2}{225} = 1$$

$$\frac{x^2}{\left(\frac{15}{5}\right)^2} - \frac{y^2}{\left(\frac{15}{6}\right)^2} = 1$$

$$\frac{x^2}{3^2} - \frac{y^2}{\left(\frac{5}{2}\right)^2} = 1$$

The obtained equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Where, $a = 3$ and $b = 5/2$

Eccentricity is given by:

$$\begin{aligned} e &= \sqrt{1 + \frac{b^2}{a^2}} \\ &= \sqrt{1 + \frac{25}{9}} \\ &= \sqrt{1 + \frac{25}{36}} \\ &= \sqrt{\frac{61}{36}} \\ &= \frac{\sqrt{61}}{6} \end{aligned}$$

Foci: The coordinates of the foci are $(\pm ae, 0)$

$$(\pm ae, 0) = \pm 3 \left(\frac{\sqrt{61}}{6} \right) = \pm \frac{\sqrt{61}}{2}$$

$$(\pm ae, 0) = \left(\pm \frac{\sqrt{61}}{2}, 0 \right)$$

The equation of directrices is given as:

$$x = \pm \frac{a}{e}$$

$$\Rightarrow x = \pm \frac{3}{\frac{\sqrt{61}}{6}}$$

$$\Rightarrow x = \pm \frac{18}{\sqrt{61}}$$

$$\Rightarrow \sqrt{61}x = \pm 18$$

$$\sqrt{61}x \mp 18 = 0$$

The length of latus-rectum is given as:

$$2b^2/a$$

$$= \frac{2 \left(\frac{5}{2}\right)^2}{3}$$

$$= \frac{2 \times \frac{25}{4}}{3}$$

$$= \frac{25}{6}$$

$\therefore$ Transverse axis = 6, conjugate axis = 5, $e = \sqrt{61}/6$, LR = 25/6, foci = $(\pm \sqrt{61}/2, 0)$

5. Find the centre, eccentricity, foci and directions of the hyperbola

(i) $16x^2 - 9y^2 + 32x + 36y - 164 = 0$

(ii) $x^2 - y^2 + 4x = 0$

(iii) $x^2 - 3y^2 - 2x = 8$

Solution:

(i) $16x^2 - 9y^2 + 32x + 36y - 164 = 0$

Given:

The equation $\Rightarrow 16x^2 - 9y^2 + 32x + 36y - 164 = 0$

Let us find the centre, eccentricity, foci and directions of the hyperbola

By using the given equation

$$16x^2 - 9y^2 + 32x + 36y - 164 = 0$$

$$16x^2 + 32x + 16 - 9y^2 + 36y - 36 - 16 + 36 - 164 = 0$$

$$16(x^2 + 2x + 1) - 9(y^2 - 4y + 4) - 16 + 36 - 164 = 0$$

$$16(x^2 + 2x + 1) - 9(y^2 - 4y + 4) - 144 = 0$$

$$16(x + 1)^2 - 9(y - 2)^2 = 144$$

$$\frac{16(x+1)^2}{144} - \frac{9(y-2)^2}{144} = 1$$

$$\frac{(x+1)^2}{9} - \frac{(y-2)^2}{16} = 1$$

$$\frac{(x+1)^2}{3^2} - \frac{(y-2)^2}{4^2} = 1$$

Here, center of the hyperbola is (-1, 2)

So, let $x + 1 = X$ and $y - 2 = Y$

The obtained equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Where, $a = 3$ and $b = 4$

Eccentricity is given by:

$$\begin{aligned} e &= \sqrt{1 + \frac{b^2}{a^2}} \\ &= \sqrt{1 + \frac{16}{9}} \\ &= \sqrt{\frac{25}{9}} \\ &= \frac{5}{3} \end{aligned}$$

Foci: The coordinates of the foci are $(\pm ae, 0)$

$X = \pm 5$ and $Y = 0$

$x + 1 = \pm 5$ and $y - 2 = 0$

$x = \pm 5 - 1$ and $y = 2$

$x = 4, -6$ and $y = 2$

So, Foci: $(4, 2)$ $(-6, 2)$

Equation of directrix are:

$$X = \pm \frac{a}{e}$$

$$\Rightarrow X = \pm \frac{3}{\frac{5}{3}}$$

$$\Rightarrow X = \pm \frac{9}{5}$$

$$\Rightarrow 5X = \pm 9$$

$$\Rightarrow 5X \mp 9 = 0$$

$$\Rightarrow 5(x+1) \mp 9 = 0$$

$$\Rightarrow 5x + 5 \mp 9 = 0$$

$$\Rightarrow 5x + 5 - 9 = 0 \text{ and } 5x + 5 + 9 = 0$$

$$5x - 4 = 0 \text{ and } 5x + 14 = 0$$

$\therefore$ The center is $(-1, 2)$, eccentricity $(e) = 5/3$, Foci $= (4, 2) (-6, 2)$, Equation of directrix $= 5x - 4 = 0$ and $5x + 14 = 0$

(ii) $x^2 - y^2 + 4x = 0$

Given:

The equation $\Rightarrow x^2 - y^2 + 4x = 0$

Let us find the centre, eccentricity, foci and directions of the hyperbola

By using the given equation

$$x^2 - y^2 + 4x = 0$$

$$x^2 + 4x + 4 - y^2 - 4 = 0$$

$$(x+2)^2 - y^2 = 4$$

$$\frac{(x+2)^2}{4} - \frac{y^2}{4} = 1$$

$$\frac{(x+2)^2}{2^2} - \frac{y^2}{2^2} = 1$$

Here, center of the hyperbola is $(2, 0)$

So, let $x - 2 = X$

The obtained equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Where, $a = 2$ and $b = 2$

Eccentricity is given by:

$$\begin{aligned}
 e &= \sqrt{1 + \frac{b^2}{a^2}} \\
 &= \sqrt{1 + \frac{4}{4}} \\
 &= \sqrt{1 + 1} \\
 &= \sqrt{2}
 \end{aligned}$$

Foci: The coordinates of the foci are $(\pm ae, 0)$

$$X = \pm 2\sqrt{2} \text{ and } Y = 0$$

$$X + 2 = \pm 2\sqrt{2} \text{ and } Y = 0$$

$$X = \pm 2\sqrt{2} - 2 \text{ and } Y = 0$$

$$\text{So, Foci} = (\pm 2\sqrt{2} - 2, 0)$$

Equation of directrix are:

$$X = \pm \frac{a}{e}$$

$$\Rightarrow X = \pm \frac{2}{\sqrt{2}}$$

$$\Rightarrow X = \pm \frac{2}{\sqrt{2}}$$

$$\Rightarrow X = \pm \sqrt{2}$$

$$\Rightarrow X \mp \sqrt{2} = 0$$

$$\Rightarrow x + 2 \mp \sqrt{2} = 0$$

$$x + 2 - \sqrt{2} = 0 \text{ and } x + 2 + \sqrt{2} = 0$$

$\therefore$ The center is $(-2, 0)$, eccentricity $(e) = \sqrt{2}$, Foci $= (-2 \pm 2\sqrt{2}, 0)$, Equation of directrix =

$$x + 2 = \pm \sqrt{2}$$

$$\text{(iii) } x^2 - 3y^2 - 2x = 8$$

Given:

$$\text{The equation} \Rightarrow x^2 - 3y^2 - 2x = 8$$

Let us find the centre, eccentricity, foci and directions of the hyperbola

By using the given equation

$$x^2 - 3y^2 - 2x = 8$$

$$x^2 - 2x + 1 - 3y^2 - 1 = 8$$

$$(x - 1)^2 - 3y^2 = 9$$

$$\frac{(x-1)^2}{9} - \frac{3y^2}{9} = 1$$

$$\frac{(x-1)^2}{3^2} - \frac{y^2}{(\sqrt{3})^2} = 1$$

Here, center of the hyperbola is (1, 0)

So, let $x - 1 = X$

The obtained equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Where, $a = 3$ and $b = \sqrt{3}$

Eccentricity is given by:

$$\begin{aligned} e &= \sqrt{1 + \frac{b^2}{a^2}} \\ &= \sqrt{1 + \frac{3}{9}} \\ &= \sqrt{1 + \frac{1}{3}} \\ &= \sqrt{\frac{4}{3}} \\ &= \frac{2}{\sqrt{3}} \\ &= \frac{2 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} \\ &= \frac{2\sqrt{3}}{3} \end{aligned}$$

Foci: The coordinates of the foci are $(\pm ae, 0)$

$$X = \pm 2\sqrt{3} \text{ and } Y = 0$$

$$X - 1 = \pm 2\sqrt{3} \text{ and } Y = 0$$

$$X = \pm 2\sqrt{3} + 1 \text{ and } Y = 0$$

$$\text{So, Foci} = (1 \pm 2\sqrt{3}, 0)$$

Equation of directrix are:

$$X = \pm \frac{a}{e}$$

$$\Rightarrow X = \pm \frac{3}{\frac{2\sqrt{3}}{3}}$$

$$\Rightarrow X = \pm \frac{9}{2\sqrt{3}}$$

$$X = \pm \frac{9}{2\sqrt{3}} + 1$$

$$X = \pm \frac{9}{2\sqrt{3}}$$

$\therefore$ The center is (1, 0), eccentricity (e) = $2\sqrt{3}/3$, Foci = $(1 \pm 2\sqrt{3}, 0)$, Equation of directrix = $X = 1 \pm 9/2\sqrt{3}$

6. Find the equation of the hyperbola, referred to its principal axes as axes of coordinates, in the following cases:

(i) the distance between the foci = 16 and eccentricity = $\sqrt{2}$

(ii) conjugate axis is 5 and the distance between foci = 13

(iii) conjugate axis is 7 and passes through the point (3, -2)

Solution:

(i) the distance between the foci = 16 and eccentricity = $\sqrt{2}$

Given:

Distance between the foci = 16

Eccentricity = $\sqrt{2}$

Let us compare with the equation of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Distance between the foci is $2ae$ and $b^2 = a^2(e^2 - 1)$

So,

$$2ae = 16$$

$$ae = 16/2$$

$$a\sqrt{2} = 8$$

$$a = 8/\sqrt{2}$$

$$a^2 = 64/2$$

$$= 32$$

We know that, $b^2 = a^2(e^2 - 1)$

$$\text{So, } b^2 = 32 [(\sqrt{2})^2 - 1]$$

$$= 32 (2 - 1)$$

$$= 32$$

The Equation of hyperbola is given as

$$\frac{x^2}{32} - \frac{y^2}{32} = 1$$

$$x^2 - y^2 = 32$$

∴ The Equation of hyperbola is $x^2 - y^2 = 32$

(ii) conjugate axis is 5 and the distance between foci = 13

Given:

Conjugate axis = 5

Distance between foci = 13

Let us compare with the equation of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Distance between the foci is $2ae$ and $b^2 = a^2(e^2 - 1)$

Length of conjugate axis is $2b$

So,

$$2b = 5$$

$$b = 5/2$$

$$b^2 = 25/4$$

We know that, $2ae = 13$

$$ae = 13/2$$

$$a^2e^2 = 169/4$$

$$b^2 = a^2(e^2 - 1)$$

$$b^2 = a^2e^2 - a^2$$

$$25/4 = 169/4 - a^2$$

$$a^2 = 169/4 - 25/4$$

$$= 144/4$$

$$= 36$$

The Equation of hyperbola is given as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{x^2}{36} - \frac{y^2}{\frac{25}{4}} = 1$$

$$\Rightarrow \frac{x^2}{36} - \frac{4y^2}{25} = 1$$

$$\Rightarrow \frac{25x^2 - 144y^2}{900} = 1$$

$$\Rightarrow 25x^2 - 144y^2 = 900$$

$\therefore$ The Equation of hyperbola is $25x^2 - 144y^2 = 900$

(iii) conjugate axis is 7 and passes through the point (3, -2)

Given:

Conjugate axis = 7

Passes through the point (3, -2)

Conjugate axis is $2b$

So,

$$2b = 7$$

$$b = 7/2$$

$$b^2 = 49/4$$

The Equation of hyperbola is given as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Since it passes through points (3, -2)

$$\Rightarrow \frac{(3)^2}{a^2} - \frac{(-2)^2}{\frac{49}{4}} = 1$$

$$\Rightarrow \frac{9}{a^2} - \frac{4(4)}{49} = 1$$

$$\Rightarrow \frac{9}{a^2} - \frac{16}{49} = 1$$

$$\Rightarrow \frac{9}{a^2} = 1 + \frac{16}{49}$$

$$\Rightarrow \frac{9}{a^2} = \frac{49 + 16}{49}$$

$$\Rightarrow \frac{9}{a^2} = \frac{65}{49}$$

$$\Rightarrow a^2 = \frac{49}{65} \times 9$$

$$a^2 = 441/65$$

The equation of hyperbola is given as:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$a^2 = 441/65 \text{ and } b^2 = 49/4$$

$$\Rightarrow \frac{x^2}{\frac{441}{65}} - \frac{y^2}{\frac{49}{4}} = 1$$

$$\Rightarrow \frac{65x^2}{441} - \frac{4y^2}{49} = 1$$

$$\Rightarrow \frac{65x^2 - 36y^2}{441} = 1$$

$$\Rightarrow 65x^2 - 36y^2 = 441$$

$\therefore$ The Equation of hyperbola is $65x^2 - 36y^2 = 441$