

RD Sharma Solutions for Class 11 Maths Chapter 21 Some Special Series

Find the sum of the following series to n terms:

1. $1^3 + 3^3 + 5^3 + 7^3 + \dots$

Solution:

Let T_n be the nth term of the given series.

We have:

$$T_n = [1 + (n - 1)2]^3$$

$$= (2n - 1)^3$$

$$= (2n)^3 - 3(2n)^2 \cdot 1 + 3 \cdot 1^2 \cdot 2n - 1^3 \text{ [Since, } (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3]$$

$$= 8n^3 - 12n^2 + 6n - 1$$

Now, let S_n be the sum of n terms of the given series.

We have:

$$\begin{aligned} S_n &= \sum_{k=1}^n T_k \\ &= \sum_{k=1}^n [2k - 1]^3 \\ &= \sum_{k=1}^n [8k^3 - 12k^2 + 6k - 1] \\ &= \sum_{k=1}^n [8k^3 - 12k^2 + 6k - 1] \\ &= \sum_{k=1}^n [8k^3 - 12k^2 + 6k - 1] \\ &= 8 \sum_{k=1}^n k^3 - \sum_{k=1}^n 12k^2 + \sum_{k=1}^n 6k - \sum_{k=1}^n 1 \\ &= \frac{8n^2(n+1)^2}{4} - n - \frac{12n(n+1)(2n+1)}{6} + \frac{6n(n+1)}{2} \end{aligned}$$

Upon simplification we get,

$$= 2n^2(n+1)^2 - n - 2n(n+1)(2n+1) + 3n(n+1)$$

$$= n(n+1)[2n(n+1) - 2(2n+1) + 3] - n$$

$$= n(n+1)[2n^2 - 2n + 1] - n$$

$$= n[2n^3 - 2n^2 + n + 2n^2 - 2n + 1 - 1]$$

$$= n[2n^3 - n]$$

$$= n^2[2n^2 - 1]$$

$$\therefore \text{The sum of the series is } n^2[2n^2 - 1]$$

2. $2^3 + 4^3 + 6^3 + 8^3 + \dots$

Solution:

Let T_n be the n th term of the given series.

We have:

$$T_n = (2n)^3$$

$$= 8n^3$$

Now, let S_n be the sum of n terms of the given series.

We have:

$$\begin{aligned} S_n &= \sum_{k=1}^n 8k^3 \\ &= 8 \sum_{k=1}^n k^3 \\ &= 8 \left[\frac{n(n+1)}{2} \right]^2 \\ &= 8 \times \frac{n^2(n+1)^2}{4} \\ &= 2n^2(n+1)^2 \\ &= 2\{n(n+1)\}^2 \end{aligned}$$

$\therefore$ The sum of the series is $2\{n(n+1)\}^2$

3. $1.2.5 + 2.3.6 + 3.4.7 + \dots$

Solution:

Let T_n be the n th term of the given series.

We have:

$$T_n = n(n+1)(n+4)$$

$$= n(n^2 + 5n + 4)$$

$$= n^3 + 5n^2 + 4n$$

Now, let S_n be the sum of n terms of the given series.

We have:

$$\begin{aligned}
 S_n &= \sum_{k=1}^n T_k \\
 &= \sum_{k=1}^n k^3 + \sum_{k=1}^n 5k^2 + \sum_{k=1}^n 4k \\
 &= \sum_{k=1}^n k^3 + 5 \sum_{k=1}^n k^2 + 4 \sum_{k=1}^n k
 \end{aligned}$$

Upon simplification we get,

$$\begin{aligned}
 &= \frac{n^2(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} \\
 &= \frac{n^2(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{6} + 2n(n+1) \\
 &= \frac{n(n+1)}{2} \left(\frac{n(n+1)}{2} + \frac{5(2n+1)}{3} + 4 \right) \\
 &= \frac{n(n+1)}{2} \left(\frac{n^2+n}{2} + \frac{10n+5}{3} + 4 \right) \\
 &= \frac{n(n+1)}{2} \left(\frac{3n^2+3n+20n+10+24}{6} \right) \\
 &= \frac{n}{12} (n+1)(3n^2 + 23n + 34) \\
 &= \frac{n(n+1)}{2} \left(\frac{3n^2+3n+20n+10+24}{6} \right) \\
 &= \frac{n}{12} (n+1)(3n^2 + 23n + 34)
 \end{aligned}$$

∴ The sum of the series is

$$= \frac{n}{12} (n+1)(3n^2 + 23n + 34)$$

4. 1.2.4 + 2.3.7 + 3.4.10 + ... to n terms.

Solution:

Let T_n be the nth term of the given series.

We have:

$$\begin{aligned}
 T_n &= n(n+1)(3n+1) \\
 &= n(3n^2 + 4n + 1) \\
 &= 3n^3 + 4n^2 + n
 \end{aligned}$$

Now, let S_n be the sum of n terms of the given series.

We have:

$$\begin{aligned}
S_n &= \sum_{k=1}^n T_k \\
&= \sum_{k=1}^n 3k^3 + \sum_{k=1}^n 4k^2 + \sum_{k=1}^n k \\
&= 3 \sum_{k=1}^n k^3 + 4 \sum_{k=1}^n k^2 + \sum_{k=1}^n k
\end{aligned}$$

Upon simplification we get,

$$\begin{aligned}
&= \frac{3n^2(n+1)^2}{4} + \frac{4n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\
&= \frac{3n^2(n+1)^2}{4} + \frac{2n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2} \\
&= \frac{n(n+1)}{2} \left(\frac{3n(n+1)}{2} + \frac{4(2n+1)}{3} + 1 \right) \\
&= \frac{n(n+1)}{2} \left(\frac{3n^2+3n}{2} + \frac{8n+4}{3} + 1 \right) \\
&= \frac{n(n+1)}{2} \left(\frac{9n^2+9n+16n+8+6}{6} \right) \\
&= \frac{n}{12} (n+1)(9n^2 + 25n + 14)
\end{aligned}$$

∴ The sum of the series is

$$= \frac{n}{12} (n+1)(9n^2 + 25n + 14)$$

5. $1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4) + \dots$ to n terms

Solution:

Let T_n be the n th term of the given series.

We have:

$$T_n = n(n+1)/2$$

$$= (n^2 + n)/2$$

Now, let S_n be the sum of n terms of the given series.

We have:

$$\begin{aligned}
S_n &= \sum_{k=1}^n T_k \\
&= \sum_{k=1}^n \left(\frac{k^2+k}{2} \right) \\
&= \frac{1}{2} \sum_{k=1}^n (k^2 + k) \\
&= \frac{1}{2} \left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right] \\
&= \frac{n(n+1)}{4} \left(\frac{2n+1}{3} + 1 \right) \\
&= \frac{n(n+1)}{4} \left(\frac{2n+4}{3} \right) \\
&= \frac{n(n+1)(2n+4)}{12} \\
&= \frac{n(n+1)(n+2)}{6}
\end{aligned}$$

∴ The sum of the series is $[n(n+1)(n+2)]/6$

EXERCISE 21.2 PAGE NO: 21.18

Sum the following series to n terms:

1. $3 + 5 + 9 + 15 + 23 + \dots$

Solution:

Let T_n be the nth term and S_n be the sum to n terms of the given series.

We have,

$$S_n = 3 + 5 + 9 + 15 + 23 + \dots + T_{n-1} + T_n \dots (1)$$

Equation (1) can be rewritten as:

$$S_n = 3 + 5 + 9 + 15 + 23 + \dots + T_{n-1} + T_n \dots (2)$$

By subtracting (2) from (1) we get

$$S_n = 3 + 5 + 9 + 15 + 23 + \dots + T_{n-1} + T_n$$

$$S_n = 3 + 5 + 9 + 15 + 23 + \dots + T_{n-1} + T_n$$

$$0 = 3 + [2 + 4 + 6 + 8 + \dots + (T_n - T_{n-1})] - T_n$$

The difference between the successive terms are $5-3 = 2$, $9-5 = 4$, $15-9 = 6$,

So these differences are in A.P

Now,

$$3 + \left[\frac{(n-1)}{2} \{4 + (n-2)2\} \right] - T_n = 0$$

$$3 + \left[\frac{(n-1)}{2} (2n) \right] = T_n$$

$$3 + n(n-1) = T_n$$

Now,

$$\begin{aligned} S_n &= \sum_{k=1}^n T_k \\ &= \sum_{k=1}^n \{3 + k(k-1)\} \\ &= \sum_{k=1}^n k^2 + \sum_{k=1}^n 3 - \sum_{k=1}^n k \\ &= \frac{n(n+1)(2n+1)}{6} + 3n - \frac{n(n+1)}{2} \\ &= \frac{n}{3} \left[\frac{(n+1)(2n+1)}{2} + 9 - \frac{3}{2}(n+1) \right] \\ &= \frac{n[n^2+8]}{3} \\ &= \frac{n}{3}(n^2+8) \end{aligned}$$

∴ The sum of the series is $n/3 (n^2 + 8)$

2. 2 + 5 + 10 + 17 + 26 +

Solution:

Let T_n be the n th term and S_n be the sum to n terms of the given series.

We have,

$$S_n = 2 + 5 + 10 + 17 + 26 + \dots + T_{n-1} + T_n \dots (1)$$

Equation (1) can be rewritten as:

$$S_n = 2 + 5 + 10 + 17 + 26 + \dots + T_{n-1} + T_n \dots (2)$$

By subtracting (2) from (1) we get

$$S_n = 2 + 5 + 10 + 17 + 26 + \dots + T_{n-1} + T_n$$

$$S_n = 2 + 5 + 10 + 17 + 26 + \dots + T_{n-1} + T_n$$

$$0 = 2 + [3 + 5 + 7 + 9 + \dots + (T_n - T_{n-1})] - T_n$$

The difference between the successive terms are 3, 5, 7, 9

So these differences are in A.P

Now,

$$2 + \left[\frac{(n-1)}{2} \{6 + (n-2)2\} \right] - T_n = 0$$

$$2 + [n^2 - 1] = T_n$$

$$[n^2 + 1] = T_n$$

Now,

$$\begin{aligned} S_n &= \sum_{k=1}^n T_k \\ &= \sum_{k=1}^n (k^2 + 1) \\ &= \sum_{k=1}^n k^2 + \sum_{k=1}^n 1 \\ &= \frac{n(n+1)(2n+1)}{6} + n \\ &= \frac{n(n+1)(2n+1)+6n}{6} \\ &= \frac{n(2n^2+3n+7)}{6} \\ &= \frac{n}{6}(2n^2 + 3n + 7) \end{aligned}$$

∴ The sum of the series is $n/6 (2n^2 + 3n + 7)$

3. $1 + 3 + 7 + 13 + 21 + \dots$

Solution:

Let T_n be the n th term and S_n be the sum to n terms of the given series.

We have,

$$S_n = 1 + 3 + 7 + 13 + 21 + \dots + T_{n-1} + T_n \dots (1)$$

Equation (1) can be rewritten as:

$$S_n = 1 + 3 + 7 + 13 + 21 + \dots + T_{n-1} + T_n \dots (2)$$

By subtracting (2) from (1) we get

$$S_n = 1 + 3 + 7 + 13 + 21 + \dots + T_{n-1} + T_n$$

$$S_n = 1 + 3 + 7 + 13 + 21 + \dots + T_{n-1} + T_n$$

$$0 = 1 + [2 + 4 + 6 + 8 + \dots + (T_n - T_{n-1})] - T_n$$

The difference between the successive terms are 2, 4, 6, 8

So these differences are in A.P

Now,

$$1 + \left[\frac{(n-1)}{2} \{4 + (n-2)2\} \right] - T_n = 0$$

$$1 + [n^2 - n] = T_n$$

$$[n^2 - n + 1] = T_n$$

Now,

$$\begin{aligned} S_n &= \sum_{k=1}^n T_k \\ &= \sum_{k=1}^n (k^2 - k + 1) \\ &= \sum_{k=1}^n k^2 + \sum_{k=1}^n 1 - \sum_{k=1}^n k \\ &= \frac{n(n+1)(2n+1)}{6} + n - \frac{n(n+1)}{2} \\ &= \frac{n(n+1)}{2} \left(\frac{2n-2}{3} \right) + n \\ &= n \left(\frac{n^2-1+3}{3} \right) \\ &= \frac{n}{3}(n^2 + 2) \end{aligned}$$

∴ The sum of the series is $n/3 (n^2 + 2)$

4. 3 + 7 + 14 + 24 + 37 + ...

Solution:

Let T_n be the n th term and S_n be the sum to n terms of the given series.

We have,

$$S_n = 3 + 7 + 14 + 24 + 37 + \dots + T_{n-1} + T_n \dots (1)$$

Equation (1) can be rewritten as:

$$S_n = 3 + 7 + 14 + 24 + 37 + \dots + T_{n-1} + T_n \dots (2)$$

By subtracting (2) from (1) we get

$$S_n = 3 + 7 + 14 + 24 + 37 + \dots + T_{n-1} + T_n$$

$$S_n = 3 + 7 + 14 + 24 + 37 + \dots + T_{n-1} + T_n$$

$$0 = 3 + [4 + 7 + 10 + 13 + \dots + (T_n - T_{n-1})] - T_n$$

The difference between the successive terms are 4, 7, 10, 13

So these differences are in A.P

Now,

$$3 + \left[\frac{(n-1)}{2} \{8 + (n-2)3\} \right] - T_n = 0$$

$$3 + \left[\frac{(n-1)}{2} (3n + 2) \right] - T_n = 0$$

$$\left[\frac{3n^2 - n + 4}{2} \right] = T_n$$

$$\left[\frac{3}{2}n^2 - \frac{n}{2} + 2 \right] = T_n$$

Now,

$$\begin{aligned} S_n &= \sum_{k=1}^n T_k \\ &= \sum_{k=1}^n \left(\frac{3}{2}k^2 - \frac{k}{2} + 2 \right) \\ &= \frac{3}{2} \sum_{k=1}^n k^2 + \sum_{k=1}^n 2 - \frac{1}{2} \sum_{k=1}^n k \\ &= \frac{n(n+1)(2n+1)}{4} + 2n - \frac{n(n+1)}{4} \\ &= \frac{n(n+1)(2n)+8n}{4} \\ &= \frac{(n+1)(2n^2)+8n}{4} \\ &= \frac{n}{2} [n(n+1) + 4] \\ &= \frac{n}{2} [n^2 + n + 4] \end{aligned}$$

∴ The sum of the series is $n/2 [n^2 + n + 4]$

5. 1 + 3 + 6 + 10 + 15 + ...

Solution:

Let T_n be the n th term and S_n be the sum to n terms of the given series.

We have,

$$S_n = 1 + 3 + 6 + 10 + 15 + \dots + T_{n-1} + T_n \dots (1)$$

Equation (1) can be rewritten as:

$$S_n = 1 + 3 + 6 + 10 + 15 + \dots + T_{n-1} + T_n \dots (2)$$

By subtracting (2) from (1) we get

$$S_n = 1 + 3 + 6 + 10 + 15 + \dots + T_{n-1} + T_n$$

$$S_n = 1 + 3 + 6 + 10 + 15 + \dots + T_{n-1} + T_n$$

$$0 = 1 + [2 + 3 + 4 + 5 + \dots + (T_n - T_{n-1})] - T_n$$

The difference between the successive terms are 2, 3, 4, 5

So these differences are in A.P

Now,

$$1 + \left[\frac{(n-1)}{2} (4 + (n-2)1) \right] - T_n = 0$$

$$1 + \left[\frac{(n-1)}{2} (n+2) \right] - T_n = 0$$

$$\left[\frac{n^2+n}{2} \right] = T_n$$

Now,

$$\begin{aligned} S_n &= \sum_{k=1}^n T_k \\ &= \sum_{k=1}^n \left(\frac{k^2+k}{2} \right) \\ &= \frac{1}{2} \sum_{k=1}^n k^2 + \frac{1}{2} \sum_{k=1}^n k \\ &= \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4} \\ &= \frac{n(n+1)}{4} \left(\frac{2n+1}{3} + 1 \right) \\ &= \frac{n(n+1)}{4} \left(\frac{2n+4}{3} \right) \\ &= \frac{n(n+1)}{2} \left(\frac{n+2}{3} \right) \\ &= \frac{n(n+1)(n+2)}{6} \\ &= \frac{n}{6} (n+1)(n+2) \end{aligned}$$

∴ The sum of the series is $\frac{n}{6} (n+1)(n+2)$