

RD Sharma Solutions For Class 12 Chapter 19 Indefinite Integrals

1. Evaluate the following integrals:

(i) $\int x^4 dx$

Solution:

Given

$$\int x^4 dx$$

Now we have integrate the given function

$$\begin{aligned} &= \frac{x^{4+1}}{4+1} + C \\ &= \frac{x^5}{5} + C \end{aligned}$$

(ii) $\int x^{5/4} dx$

Solution:

Given

$$\int x^{5/4} dx$$

Now we have to integrate the given function

$$= \frac{x^{\frac{5}{4} + 1}}{\frac{5}{4} + 1} + C$$

On simplifying, we get

$$= \frac{4}{9} x^{\frac{9}{4}} + C$$

$$(iii) \int \frac{1}{x^5} dx$$

Solution:

Given

$$\int \frac{1}{x^5} dx$$

We can write given question as

$$\int x^{-5} dx$$

Now by integrating, we get

$$= \frac{x^{-5+1}}{-5+1} + C$$

$$= -\frac{1}{4}x^{-4} + C$$

On simplifying we get

$$= -\frac{1}{4x^4} + C$$

$$(iv) \int \frac{1}{x^{\frac{3}{2}}} dx$$

Solution:

Given

$$\int \frac{1}{x^{\frac{3}{2}}} dx$$

Given equation can be written as

$$\int x^{-\frac{3}{2}} dx$$

Now by integrating the above equation we get

$$= \left[\frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} \right] + C$$

$$= \left[\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} \right] + C$$

On simplifying we get

$$= -\frac{2}{\sqrt{x}} + C$$

$$= \left[\frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} \right] + C$$

$$= \left[\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} \right] + C$$

On simplifying we get

$$= -\frac{2}{\sqrt{x}} + C$$

$$(v) \int 3^x dx$$

Solution:

Given

$$\int 3^x dx$$

We know that

$$\int a^x dx = \frac{a^x}{\log_e a} + c$$

Now by integrating the given equation by using above integration formulae we get

$$\int 3^x dx = \frac{3^x}{\log 3} + c$$

(vi) $\int \frac{1}{\sqrt[3]{x^2}} dx$

Solution:

Given

$$\int \frac{1}{\sqrt[3]{x^2}} dx$$

now above equation can be written as

$$= \int \frac{dx}{x^{2/3}}$$

$$= \int x^{-2/3} dx$$

Now by integrating the above equation we get

$$= \frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} + C$$

On simplifying

$$= 3x^{\frac{1}{3}} + C$$

$$(vii) \int 3^{2 \log_3 x} dx$$

Solution:

Given

$$\int 3^{2 \log_3 x} dx$$

Given equation can be written as

$$= \int 3 \log_3 x^2 dx$$

On simplifying we get

$$= \int x^2 dx$$

Now by integrating the above equation we get

$$= \frac{x^3}{3} + C$$

Now by integrating the above equation we get

$$= \frac{x^3}{3} + C$$

$$(viii) \int \log_x x dx$$

Solution:

Given

$$\int \log_x x \, dx$$

Given equation can be written as

$$= \int 1 \cdot dx$$

By integrating we get

$$= x + C$$

2. Evaluate:

$$(i) \int \sqrt{\frac{1 + \cos 2x}{2}} \, dx$$

Solution:

Given

$$\int \sqrt{\frac{1 + \cos 2x}{2}} \, dx$$

Given equation can be written as

$$\int \sqrt{\frac{2 \cos^2 x}{2}} \, dx \quad [\because 1 + \cos 2A = 2 \cos^2 A]$$

On simplifying, we get

$$= \int \cos x \, dx$$

On integrating

$$= \sin x + C$$

$$= \int \cos x \, dx$$

On integrating

$$= \sin x + C$$

$$(ii) \int \sqrt{\frac{1 - \cos 2x}{2}} \, dx$$

Solution:

Given

$$\int \sqrt{\frac{1 - \cos 2x}{2}} \, dx$$

Given equation can be written as

$$= \int \sqrt{\frac{2 \sin^2 x}{2}} \, dx \quad [\because 1 - \cos 2x = 2 \sin^2 x]$$

On simplifying we get

$$= \int \sin x \, dx$$

On integrating

$$= -\cos x + C$$

3. Evaluate:

$$\int \frac{e^{6 \log_e x} - e^{5 \log_e x}}{e^{4 \log_e x} - e^{3 \log_e x}} dx$$

Solution:

Given

$$\begin{aligned} & \int \frac{e^{6 \log_e x} - e^{5 \log_e x}}{e^{4 \log_e x} - e^{3 \log_e x}} dx \\ &= \int \left(\frac{e^{\log x^6} - e^{\log x^5}}{e^{\log x^4} - e^{\log x^3}} \right) dx \end{aligned}$$

Above equation can be written as

$$\begin{aligned} &= \int \left(\frac{x^6 - x^5}{x^4 - x^3} \right) dx \\ &= \int \frac{x^5}{x^3} dx \\ &= \int x^2 dx \end{aligned}$$

Now by integrating we get

$$= \frac{x^3}{3} + C$$

Exercise 19.2 Page No: 19.14

Evaluate the following integrals (1 – 44):

$$1. \int (3x\sqrt{x} + 4\sqrt{x} + 5) dx$$

Solution:

Given

$$\int (3x\sqrt{x} + 4\sqrt{x} + 5) dx$$

By Splitting, we get,

$$\Rightarrow \int ((3x\sqrt{x})dx + (4\sqrt{x})dx + 5dx)$$

$$\Rightarrow \int 3x\sqrt{x}dx + \int 4\sqrt{x}dx + \int 5dx$$

$$\Rightarrow \int 3x^{\frac{3}{2}}dx + \int 4x^{\frac{1}{2}}dx + \int 5dx$$

By using the formula, $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$\Rightarrow \frac{3x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{4x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \int 5dx$$

We know that

$$\int kdx = kx + c$$

$$\Rightarrow \frac{3x^{\frac{5}{2}}}{5/2} + \frac{4x^{\frac{3}{2}}}{3/2} + 5x + c$$

$$\Rightarrow \frac{6}{5}x^{\frac{5}{2}} + \frac{8}{3}x^{3/2} + 5x + c$$

$$2. \int \left(2^x + \frac{5}{x} - \frac{1}{x^{\frac{1}{3}}} \right) dx$$

Solution:

Given

$$\int \left(2^x + \frac{5}{x} - \frac{1}{x^{1/3}} \right) dx$$

By splitting given equation we get, we get,

$$\Rightarrow \int 2^x dx + \int \left(\frac{5}{x} \right) dx - \int \frac{1}{x^{1/3}} dx$$

By using the formula,

$$\int a^x dx = \frac{a^x}{\log a}$$

$$\Rightarrow \frac{2^x}{\log 2} + 5 \int \left(\frac{1}{x} \right) dx - \int x^{-1/3} dx$$

Again by using the formula,

$$\int \left(\frac{1}{x} \right) dx = \log x$$

$$\Rightarrow \frac{2^x}{\log 2} + 5 \log x - \int x^{-1/3} dx$$

By using the below formula, we get

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \frac{2^x}{\log 2} + 5 \log x - \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}$$

$$\Rightarrow \frac{2^x}{\log 2} + 5 \log x - \frac{x^{\frac{2}{3}}}{2/3}$$

On simplifying we get

$$\Rightarrow \frac{2^x}{\log 2} + 5 \log x - \frac{3}{2} x^{2/3} + c$$

$$\Rightarrow \frac{2^x}{\log 2} + 5 \log x - \frac{3}{2} x^{2/3} + c$$

$$3. \int \{ \sqrt{x}(ax^2 + bx + c) \} dx$$

Solution:

Given

$$\int \{\sqrt{x}(ax^2 + bx + c)\}dx$$

Now by multiplying we get

$$\Rightarrow \int (\sqrt{x}ax^2 + \sqrt{x}bx + \sqrt{x}c) dx$$

By splitting the given equation, we get,

$$\Rightarrow a \int x^2 \times x^{\frac{1}{2}} dx + b \int x^1 \times x^{\frac{1}{2}} dx + c \int x^{1/2} dx$$

$$\Rightarrow a \int x^{\frac{5}{2}} dx + b \int x^{\frac{3}{2}} dx + c \int x^{\frac{1}{2}} dx$$

By using the formula

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

We get

$$\Rightarrow \frac{ax^{\frac{5}{2}+1}}{\frac{5}{2}+1} + \frac{bx^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{cx^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

On simplifying

$$\Rightarrow \frac{ax^{\frac{7}{2}}}{7/2} + \frac{bx^{\frac{5}{2}}}{5/2} + \frac{cx^{\frac{3}{2}}}{3/2} + c$$

$$4. \int (2 - 3x)(3 + 2x)(1 - 2x) dx$$

Solution:

Given,

$$\int (2 - 3x)(3 + 2x)(1 - 2x) dx$$

$$= \int (6 + 4x - 9x - 6x^2)(1 - 2x) dx$$

$$= \int (6 - 5x - 6x^2)(1 - 2x) dx$$

$$= \int (6 - 5x - 6x^2 - 12x + 10x^2 + 12x^3) dx$$

$$= \int (6 - 17x + 4x^2 + 12x^3) dx$$

Upon splitting the above, we have

$$= \int 6 \, dx - \int 17x \, dx + \int 4x^2 \, dx + \int 12x^3 \, dx$$

On integrating using formula,

$$\int x^n \, dx = x^{n+1}/n+1$$

we get

$$= 6x - 17/(1+1) x^{1+1} + 4/(2+1) x^{2+1} + 12/(3+1) x^{3+1} + c$$

$$= 6x - 17x^2/2 + 4x^3/3 + 3x^4 + c$$

$$5. \int \left(\frac{m}{x} + \frac{x}{m} + m^x + x^m + mx \right) dx$$

Solution:

Given

$$\int \left(\frac{m}{x} + \frac{x}{m} + m^x + x^m + mx \right) dx$$

By Splitting, we get,

$$\Rightarrow \int \frac{m}{x} dx + \int \frac{x}{m} dx + \int x^m dx + \int m^x dx + \int mx dx$$

We have

$$\Rightarrow \int \frac{m}{x} dx + \int \frac{x}{m} dx + \int x^m dx + \int m^x dx + \int mx dx$$

We have

$$\int \frac{1}{x} dx = \log x + c$$

By applying the above formula, we get

$$\Rightarrow m \log x + \frac{1}{m} \int x dx + \int x^m dx + \int m^x dx + \int mx dx$$

By using this, we have

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow m \log x + \frac{\frac{1}{m} x^{1+1}}{1+1} + \frac{x^{m+1}}{m+1} + \int m^x dx + \frac{mx^{1+1}}{1+1}$$

By using the formula,

$$\int a^x dx = \frac{a^x}{\log a}$$

$$\Rightarrow m \log x + \frac{\frac{1}{m} x^2}{2} + \frac{x^{m+1}}{m+1} + \frac{m^x}{\log m} + \frac{mx^2}{2} + c$$

$$6. \int \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right)^2 dx$$

Solution:

$$\left(\sqrt{x} - \frac{1}{\sqrt{x}} \right)^2 dx$$

By applying $(a - b)^2 = a^2 - 2ab + b^2$ we get

$$\Rightarrow \int \left((\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}} \right)^2 - 2(\sqrt{x}) \left(\frac{1}{\sqrt{x}} \right) \right) dx$$

After computing or simplifying, we get

$$\Rightarrow \int \left(x + \frac{1}{x} - 2 \right) dx$$

By splitting the above equation, we get,

$$\Rightarrow \int x dx + \int \frac{1}{x} dx - 2 \int dx$$

Now integrate by using standard integration formulae, we get

$$\begin{aligned} &\Rightarrow \frac{x^{1+1}}{1+1} + \log x - 2x + c \\ &= \frac{1}{2} x^2 + \log |x| - 2x + c \end{aligned}$$

$$7. \int \frac{(1+x)^3}{\sqrt{x}} dx$$

Solution:

Given

$$\int \frac{(1+x)^3}{\sqrt{x}} dx$$

Now by applying this formula $(a+b)^3 = a^3 + b^3 + 3ab^2 + 3a^2b$ we get

$$\Rightarrow \int \frac{1+x^3+3x^2 \times 1 + 3 \times 1^2 \times x}{\sqrt{x}} dx$$

$$\Rightarrow \int \frac{1+x^3+3x^2+3x}{\sqrt{x}} dx$$

By splitting the above equation, we get,

$$\Rightarrow \int \frac{1}{\sqrt{x}} dx + \int \frac{x^3}{\sqrt{x}} dx + \int \frac{3x^2}{\sqrt{x}} dx + \int \frac{3x}{\sqrt{x}} dx$$

$$\Rightarrow \int x^{-\frac{1}{2}} dx + \int x^3 \times x^{-\frac{1}{2}} dx + \int 3x^2 \times x^{-\frac{1}{2}} dx + \int 3x \times x^{-\frac{1}{2}} dx$$

$$\Rightarrow \int x^{-\frac{1}{2}} dx + \int x^{\frac{5}{2}} dx + 3 \int x^{\frac{3}{2}} dx + 3 \int x^{\frac{1}{2}} dx$$

Again we have formula,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

By applying the above formula we get

$$\Rightarrow \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + 3 \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$\Rightarrow \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{3x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= 2\sqrt{x} + \frac{2}{7}x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + \frac{6}{5}x^{\frac{5}{2}} + C$$

$$8. \int \left\{ x^2 + e^{\log x} + \left(\frac{e}{2} \right)^x \right\} dx$$

Solution:

Given

$$\int \left\{ x^2 + e^{\log x} + \left(\frac{e}{2} \right)^x \right\} dx$$

By splitting the above equation, we get,

$$\Rightarrow \int x^2 dx + \int e^{\log x} dx + \int \left(\frac{e}{2} \right)^x dx$$

By applying formula,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

We get

$$\Rightarrow \frac{x^{2+1}}{2+1} + \int e^{\log_e x} dx + \int \left(\frac{e}{2} \right)^x dx$$

$$\Rightarrow \frac{x^3}{3} + \int x \, dx + \frac{1}{\log\left(\frac{e}{2}\right)} \left(\frac{e}{2}\right)^x$$

$$\Rightarrow \frac{x^3}{3} + \int x \, dx + \frac{1}{\log\left(\frac{e}{2}\right)} \left(\frac{e}{2}\right)^x$$

Integrating and simplifying we get

$$\Rightarrow \frac{x^3}{3} + \frac{x^2}{2} + \frac{\left(\frac{e}{2}\right)^x}{\log\left(\frac{e}{2}\right)} + c$$

9. $\int (x^e + e^x + e^e) \, dx$

Solution:

$$\int (x^e + e^x + e^e) dx$$

By splitting the above equation, we get,

$$\Rightarrow \int x^e dx + \int e^x dx + \int e^e dx$$

By using the below formula,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

We can write as

$$\Rightarrow \frac{x^{e+1}}{e+1} + \int e^x dx + \int e^e dx$$

Again by applying the formula,

$$\int a^x dx = \frac{a^x}{\log a}$$

We get

$$\Rightarrow \frac{x^{e+1}}{e+1} + \frac{e^x}{\log_e e} + \int e^e dx$$

We know that,

$$\int k dx = kx + c$$

So substituting this we have

$$\Rightarrow \frac{x^{e+1}}{e+1} + \frac{e^x}{\log_e e} + \int e^e dx$$

We know that,

$$\int k dx = kx + c$$

So substituting this we have

$$\Rightarrow \frac{x^{e+1}}{e+1} + \frac{e^x}{\log_e e} + e^e x + c$$

$$10. \int \sqrt{x} \left(x^3 - \frac{2}{x} \right) dx$$

Solution:

Given

$$\int \sqrt{x} \left(x^3 - \frac{2}{x} \right) dx$$

Multiplying throughout the bracket, we get,

$$\Rightarrow \int \left(x^{\frac{1}{2}} \times x^3 - x^{\frac{1}{2}} \times \frac{2}{x} \right) dx$$

$$\Rightarrow \int \left(x^{\frac{1}{2}+3} - x^{\frac{1}{2}-1} \times 2 \right) dx$$

Again by simplifying

$$\Rightarrow \int \left(x^{\frac{7}{2}} - 2x^{-\frac{1}{2}} \right) dx$$

By multiplying,

$$\Rightarrow \int x^{\frac{7}{2}} dx - 2 \int x^{-\frac{1}{2}} dx$$

We have

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

Applying the above formula, we get

$$\Rightarrow \frac{x^{\frac{7}{2}+1}}{\frac{7}{2}+1} - 2 \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$\Rightarrow \frac{x^{\frac{9}{2}}}{\frac{9}{2}} - 2 \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$\Rightarrow \frac{2x^{\frac{9}{2}}}{9} - 4x^{\frac{1}{2}} + c$$

$$11. \int \frac{1}{\sqrt{x}} \left(1 + \frac{1}{x}\right) dx$$

Solution:

Given

$$\int \frac{1}{\sqrt{x}} \left\{ 1 + \frac{1}{x} \right\} dx$$

By multiplying $\frac{1}{\sqrt{x}}$ throughout the brackets,

$$\Rightarrow \int \left\{ \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x}} \times \frac{1}{x} \right\} dx$$

The above equation can be written as

$$\Rightarrow \int \left\{ \frac{1}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{2}}} \times \frac{1}{x} \right\} dx$$

$$\Rightarrow \int \left\{ \frac{1}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{2}+1}} \right\} dx$$

$$\Rightarrow \int \left\{ \frac{1}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{3}{2}}} \right\} dx$$

By splitting them, we get,

$$\Rightarrow \int x^{-\frac{1}{2}} dx + \int x^{-\frac{3}{2}} dx$$

We have

$$\Rightarrow \int x^{-\frac{1}{2}} dx + \int x^{-\frac{3}{2}} dx$$

We have

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

By applying the above formula and integrating, we get

$$\Rightarrow \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + c$$

$$\Rightarrow \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c$$

$$\Rightarrow 2x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} + c$$

$$14. \int \frac{(1 + \sqrt{x})^2}{\sqrt{x}} dx$$

Solution:

$$\int \frac{(1 + \sqrt{x})^2}{\sqrt{x}} dx$$

By applying $(a + b)^2 = a^2 + b^2 + 2ab$ we get

$$\Rightarrow \int \frac{(1)^2 + (\sqrt{x})^2 + 2 \times 1 \times \sqrt{x}}{\sqrt{x}} dx$$

$$\Rightarrow \int \frac{1 + x + 2\sqrt{x}}{\sqrt{x}} dx$$

By splitting the above equation, we get,

$$\Rightarrow \int \left(\frac{1}{\sqrt{x}} + \frac{x}{\sqrt{x}} + \frac{2\sqrt{x}}{\sqrt{x}} \right) dx$$

$$\Rightarrow \int x^{-\frac{1}{2}} dx + \int x \times x^{-\frac{1}{2}} dx + 2 \int dx$$

On simplifying and integrating

$$\Rightarrow \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \int x^{1-\frac{1}{2}} dx + 2x + c$$

$$\Rightarrow \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \int x^{\frac{1}{2}} dx + 2x + c$$

Now by integrating, we get

$$\Rightarrow 2x^{\frac{1}{2}} + \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 2x + c$$

$$\Rightarrow 2x^{\frac{1}{2}} + \frac{2x^{\frac{3}{2}}}{3} + 2x + c$$

$$15. \int \sqrt{x}(3-5x) dx$$

Solution:

Given

$$\int \sqrt{x}(3-5x) dx$$

By multiplying $\sqrt{x}$ throughout the bracket we get,

$$\Rightarrow \int (3\sqrt{x} - 5x\sqrt{x}) dx$$

$$\Rightarrow \int \left(3x^{\frac{1}{2}} - 5x^1 \times x^{\frac{1}{2}} \right) dx$$

$$\Rightarrow \int (3x^{\frac{1}{2}} - 5x^{1+\frac{1}{2}}) dx$$

$$\Rightarrow \int (3x^{\frac{1}{2}} - 5x^{\frac{3}{2}}) dx$$

By splitting the above equation, we get,

$$\Rightarrow 3 \int x^{\frac{1}{2}} dx - 5 \int x^{\frac{3}{2}} dx$$

By using the formula and integrating

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{5x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c$$

$$\Rightarrow \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{5x^{\frac{5}{2}}}{\frac{5}{2}} + c$$

$$\Rightarrow 2x^{\frac{3}{2}} - 2x^{\frac{5}{2}} + c$$

$$16. \int \frac{(x+1)(x-2)}{\sqrt{x}} dx$$

Solution:

Given

$$\int \frac{(x+1)(x-2)}{\sqrt{x}} dx$$

Multiplying the above equation, we get

$$\Rightarrow \int \frac{x^2 - 2x + x - 2}{\sqrt{x}} dx$$

$$\Rightarrow \int \frac{x^2 - x - 2}{\sqrt{x}} dx$$

By splitting the above equation,

$$\Rightarrow \int \frac{x^2}{\sqrt{x}} dx - \int \frac{x}{\sqrt{x}} dx - \int \frac{2}{\sqrt{x}} dx$$

$$\Rightarrow \int x^2 \times x^{-\frac{1}{2}} dx - \int x \times x^{-\frac{1}{2}} dx - 2 \int x^{-\frac{1}{2}} dx$$

$$\Rightarrow \int x^2 \times x^{-\frac{1}{2}} dx - \int x \times x^{-\frac{1}{2}} dx - 2 \int x^{-\frac{1}{2}} dx$$

$$\Rightarrow \int x^{2-\frac{1}{2}} dx - \int x^{1-\frac{1}{2}} dx - 2 \int x^{-\frac{1}{2}} dx$$

$$\Rightarrow \int x^{\frac{3}{2}} dx - \int x^{\frac{1}{2}} dx - 2 \int x^{-\frac{1}{2}} dx$$

We have the formula,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

By applying the above formula we get

$$\Rightarrow \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{2x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$\Rightarrow \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$\Rightarrow \frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + c$$

$$17. \int \frac{x^5 + x^{-2} + 2}{x^2} dx$$

Solution:

Given

$$\int \frac{x^5 + x^{-2} + 2}{x^2} dx$$

By splitting the above equation, we get,

$$\Rightarrow \int \left(\frac{x^5}{x^2} + \frac{x^{-2}}{x^2} + \frac{2}{x^2} \right) dx$$

The above equation can be written as

$$\Rightarrow \int (x^5 \times x^{-2} + x^{-2} \times x^{-2} + 2 \times x^{-2}) dx$$

On simplifying,

$$\Rightarrow \int (x^5 \times x^{-2} + x^{-2} \times x^{-2} + 2 \times x^{-2}) dx$$

On simplifying,

$$\Rightarrow \int (x^{5-2} + x^{-2-2} + 2x^{-2}) dx$$

$$\Rightarrow \int (x^3 + x^{-4} + 2x^{-2}) dx$$

Again by splitting the above equation, we get,

$$\Rightarrow \int x^3 dx + \int x^{-4} dx + 2 \int x^{-2} dx$$

By applying the formula,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

Now by integrating by using the formula,

$$\Rightarrow \frac{x^{3+1}}{3+1} + \frac{x^{-4+1}}{-4+1} + \frac{2x^{-2+1}}{-2+1} + c$$

$$\Rightarrow \frac{x^4}{4} + \frac{x^{-3}}{-3} + \frac{2x^{-1}}{-1} + c$$

$$20. \int \frac{5x^4 + 12x^3 + 7x^2}{x^2 + x} dx$$

Solution:

Given

$$\int \frac{5x^4 + 12x^3 + 7x^2}{x^2 + x} dx$$

Now split $12x^3$ into $7x^3$ and $5x^3$

$$\Rightarrow \int \frac{5x^4 + 7x^3 + 5x^3 + 7x^2}{x^2 + x} dx$$

Now common $5x^3$ from two elements $7x$ from other two elements,

Now common $5x^3$ from two elements $7x^2$ from other two elements

$$\Rightarrow \int \frac{5x^2(x+1) + 7x^2(x+1)}{x^2+x} dx$$

$$\Rightarrow \frac{\int (5x^3 + 7x^2)(x+1)}{x(x+1)} dx$$

$$\Rightarrow \int (5x^2 + 7x) dx$$

Now splitting the above equation, we get,

$$\Rightarrow \int 5x^2 dx + \int 7x dx$$

$$\Rightarrow \frac{5x^{2+1}}{2+1} + \frac{7x^{1+1}}{1+1} + c$$

$$\Rightarrow \frac{5x^3}{3} + \frac{7x^2}{2} + c$$

Exercise 19.3 Page No: 19.23

1. $\int (2x-3)^5 + \sqrt{3x+2} dx$

Solution:

$$\text{Let } I = \int (2x-3)^5 + \sqrt{3x+2}$$

Then,

$$I = \int (2x-3)^5 + (3x+2)^{\frac{1}{2}}$$

Now by integrating the above equation, we get

$$= \frac{(2x-3)^{5+1}}{2(5+1)} + \frac{(3x+2)^{\frac{1}{2}+1}}{3(\frac{1}{2}+1)}$$

$$= \frac{(2x-3)^6}{2(6)} + \frac{(3x+2)^{\frac{3}{2}}}{3(\frac{3}{2})}$$

$$= \frac{(2x-3)^6}{12} + \frac{2(3x+2)^{\frac{3}{2}}}{9}$$

$$\text{Hence, } I = \frac{(2x-3)^6}{12} + \frac{2(3x+2)^{\frac{3}{2}}}{9} + C$$

$$2. \int \frac{1}{(7x-5)^3} + \frac{1}{\sqrt{5x-4}} dx$$

Solution:

$$\text{Let } I = \int \frac{1}{(7x-5)^3} + \frac{1}{\sqrt{5x-4}} dx \text{ then,}$$

$$I = \int (7x-5)^{-3} + (5x-4)^{-\frac{1}{2}}$$

Integrating the above equation, we get

$$= \frac{(7x-5)^{-3+1}}{7(-3+1)} + \frac{(5x-4)^{-\frac{1}{2}+1}}{5(-\frac{1}{2}+1)}$$

$$= \frac{(7x-5)^{-2}}{-14} + \frac{(5x-4)^{\frac{1}{2}}}{5(\frac{1}{2})}$$

$$\text{Hence, } I = -\frac{1}{14}(7x-5)^{-2} + \frac{2}{5}\sqrt{5x-4} + C$$

$$3. \int \frac{1}{2-3x} + \frac{1}{\sqrt{3x-2}} dx$$

Solution:

$$\text{Let } I = \int \frac{1}{2-3x} + \frac{1}{\sqrt{3x-2}} dx$$

$$I = \int \frac{1}{2-3x} + \frac{1}{\sqrt{3x-2}} dx$$

$$\text{We know } \int \frac{1}{x} dx = \log|x|$$

By applying the above formula we get

$$= \frac{\log|2-3x|}{-3} + \frac{2}{3}(3x-2)^{\frac{1}{2}}$$

$$= -\frac{1}{3} \log|2x-3| + \frac{2}{3} \sqrt{3x-2} + C$$

$$4. \int \frac{x+3}{(x+1)^4} dx$$

Solution:

Let,

$$I = \int \frac{x+3}{(x+1)^4} dx$$

Splitting the above given equation

$$\begin{aligned} &= \int \frac{x+1}{(x+1)^4} dx + \int \frac{2}{(x+1)^4} dx \\ &= \int \frac{1}{(x+1)^3} dx + \int \frac{2}{(x+1)^4} dx \end{aligned}$$

The above equation can be written as

$$= \int (x+1)^{-3} dx + \int 2(x+1)^{-4} dx$$

Integrating the above equation we get

$$\begin{aligned} &= \frac{[x+1]^{-3+1}}{-3+1} + \frac{2[x+1]^{-4+1}}{-4+1} \\ &= \frac{[x+1]^{-2}}{-2} + \frac{2[x+1]^{-3}}{-3} \end{aligned}$$

$$\text{Hence, } I = -\frac{1}{2(x+1)^2} - \frac{2}{3(x+1)^3} + C$$

$$= \int \frac{x+1}{(x+1)^4} dx + \int \frac{2}{(x+1)^4} dx$$

$$= \int \frac{1}{(x+1)^3} dx + \int \frac{2}{(x+1)^4} dx$$

The above equation can be written as

$$= \int (x+1)^{-3} dx + \int 2(x+1)^{-4} dx$$

Integrating the above equation we get

$$= \frac{[x+1]^{-3+1}}{-3+1} + \frac{2(x+1)^{-4+1}}{-4+1}$$

$$= \frac{[x+1]^{-2}}{-2} + \frac{2(x+1)^{-3}}{-3}$$

$$\text{Hence, } I = -\frac{1}{2(x+1)^2} - \frac{2}{3(x+1)^3} + C$$

$$5. \int \frac{1}{\sqrt{x+1} + \sqrt{x}} dx$$

Solution:

$$\text{Let } I = \int \frac{1}{\sqrt{x+1} + \sqrt{x}} dx$$

Now multiply with the conjugate, we get

$$= \int \frac{1}{\sqrt{x+1} + \sqrt{x}} \cdot \frac{\sqrt{x+1} - \sqrt{x}}{\sqrt{x+1} - \sqrt{x}} dx$$

$$= \int \frac{\sqrt{x+1} - \sqrt{x}}{x+1-x} dx$$

On simplification we get

$$= \int \sqrt{x+1} - \sqrt{x} dx$$

The above equation can be written as

$$= \int (x+1)^{\frac{1}{2}} - x^{\frac{1}{2}}$$

On integrating we get

$$= \frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}}$$

$$\text{Hence } I = \frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{2}{3}(x)^{\frac{3}{2}} + C$$

$$6. \int \frac{1}{\sqrt{2x+3} + \sqrt{2x-3}} dx$$

Solution:

$$\text{Let } I = \int \frac{1}{\sqrt{2x+3} + \sqrt{2x-3}} dx$$

Now, multiply with the conjugate, we get

$$= \int \frac{1}{\sqrt{2x+3} + \sqrt{2x-3}} \times \frac{(\sqrt{2x+3} - \sqrt{2x-3})}{(\sqrt{2x+3} - \sqrt{2x-3})} dx$$

$$= \int \frac{(\sqrt{2x+3} - \sqrt{2x-3})}{(\sqrt{2x+3})^2 - (\sqrt{2x-3})^2} dx$$

$$= \int \frac{(\sqrt{2x+3} - \sqrt{2x-3})}{2x+3 - 2x+3} dx$$

On simplifying or computing we get

$$= \int \frac{\sqrt{2x+3}}{6} dx - \int \frac{\sqrt{2x-3}}{6} dx$$

Taking $1/6$ as common

$$= \frac{1}{6} \int (2x+3)^{\frac{1}{2}} dx - \frac{1}{6} \int (2x-3)^{\frac{1}{2}} dx$$

On integrating we get

$$= \frac{1}{6} \left(\frac{2x+3}{2} \right)^{\frac{1}{2}+1} - \frac{1}{6} \left[\frac{2x-3}{2} \right]^{\frac{1}{2}+1}$$

$$= \frac{1}{6} \left(\frac{2x+3}{2 \times \frac{3}{2}} \right)^{\frac{3}{2}} - \frac{1}{6} \left(\frac{2x-3}{2 \times \frac{3}{2}} \right)^{\frac{3}{2}}$$

$$= \frac{1}{6} \left(\frac{2x+3}{2 \times \frac{3}{2}} \right)^{\frac{3}{2}} - \frac{1}{6} \left(\frac{2x-3}{2 \times \frac{3}{2}} \right)^{\frac{3}{2}}$$

$$\text{Hence, } I = \frac{1}{18} (2x+3)^{\frac{3}{2}} - \frac{1}{18} (2x-3)^{\frac{3}{2}} + C$$

$$7. \int \frac{2x}{(2x+1)^2} dx$$

Solution:

$$\text{Let } I = \int \frac{2x}{(2x+1)^2} dx$$

Now by splitting the above equation we get

$$= \int \frac{2x+1}{(2x+1)^2} - \frac{1}{(2x+1)^2} dx$$

The above equation can be written as

$$= \int \frac{1}{(2x+1)} - (2x+1)^{-2} dx$$

On integrating we get

$$= \frac{1}{2} \log|2x+1| - \frac{(2x+1)^{-2+1}}{-2+1(2)}$$

$$= \frac{1}{2} \log|2x+1| - \frac{(2x+1)^{-1}}{-2}$$

$$\text{Hence, } I = \frac{1}{2} \log|2x+1| + \frac{1}{2(2x+1)} + C$$

$$8. \int \frac{1}{\sqrt{x+a} + \sqrt{x+b}} dx$$

Solution:

$$\text{Let } I = \int \frac{1}{\sqrt{x+a} + \sqrt{x+b}} dx$$

Now, multiply with conjugate, we get

$$= \int \frac{1}{\sqrt{x+a} + \sqrt{x+b}} \times \frac{(\sqrt{x+a} - \sqrt{x+b})}{(\sqrt{x+a} - \sqrt{x+b})} dx$$

Now, multiply with conjugate, we get

$$= \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{(\sqrt{x+a}-\sqrt{x+b})}{\sqrt{x+a}-\sqrt{x+b}} dx$$

$$= \int \frac{(\sqrt{x+a}-\sqrt{x+b})}{(\sqrt{x+a})^2 - (\sqrt{x+b})^2} dx$$

On computing, we get

$$= \int \frac{(\sqrt{x+a}-\sqrt{x+b})}{a-b} dx$$

On integrating the above equation we get

$$= \frac{1}{a-b} \left[\frac{2}{3} (x+a)^{\frac{3}{2}} - \frac{2}{3} (x+b)^{\frac{3}{2}} \right]$$

$$\text{Hence, } I = \frac{2}{3(a-b)} \left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}} \right] + C$$

9. $\int \sin x \sqrt{1 + \cos 2x} dx$

Solution:

$$\text{Let } I = \int \sin x \sqrt{(1 + \cos 2x)} dx$$

$$= \int \sin x \sqrt{(1 + \cos 2x)} dx$$

By substituting the formula, we get

$$= \int \sin x \sqrt{2 \cos^2 x} dx$$

$$= \int \sin x \sqrt{2} \cos x dx$$

$$= \sqrt{2} \int \sin x \cos x dx$$

Now, multiply and Divide by 2 we get,

$$= \frac{\sqrt{2}}{2} \int 2 \sin x \cos x dx$$

$$= \frac{\sqrt{2}}{2} \int \sin 2x dx$$

$$= \frac{\sqrt{2}}{2} \int \sin 2x dx$$

On integrating

$$= \frac{\sqrt{2}}{2} \frac{-\cos 2x}{2}$$

$$\text{Hence, } I = -\frac{1}{2\sqrt{2}} \cos 2x + C$$

Exercise 19.4 Page No: 19.30

$$1. \int \frac{x^2 + 5x + 2}{x + 2} dx$$

Solution:

Given

$$\int \frac{x^2 + 5x + 2}{x + 2} dx$$

By performing long division of the given equation we get

$$\text{Quotient} = x + 3$$

$$\text{Remainder} = -4$$

∴ We can write the above equation as

$$\Rightarrow x + 3 - \frac{4}{x+2}$$

∴ The above equation becomes

$$\Rightarrow \int x + 3 - \frac{4}{x+2} dx$$

By splitting

$$\Rightarrow \int x dx + 3 \int dx - 4 \int \frac{1}{x+2} dx$$

$$\text{We know } \int x dx = \frac{x^n}{n+1}; \int \frac{1}{x} dx = \ln x$$

$$\Rightarrow \frac{x^2}{2} + 3x - 4\ln(x+2) + c. \text{ (Where } c \text{ is some arbitrary constant)}$$

$$= \frac{x^2}{2} + 3x - 4 \log |x+2| + c$$

$$2. \int \frac{x^3}{x-2} dx$$

Solution:

Given

$$\int \frac{x^3}{x-2} dx$$

By performing long division of the given equation we get

$$\text{Quotient} = x^2 + 2x + 4$$

$$\text{Remainder} = 8$$

∴ We can write the above equation as

$$\Rightarrow x^2 + 2x + 4 + \frac{8}{x-2}$$

∴ The above equation becomes

$$\Rightarrow \int x^2 + 2x + 4 + \frac{8}{x-2} dx$$

$$\Rightarrow \int x^2 dx + 2 \int x dx + 4 \int dx + 8 \int \frac{1}{x-2} dx$$

$$\text{We know } \int x dx = \frac{x^n}{n+1}; \int \frac{1}{x} dx = \ln x$$

$$\Rightarrow \frac{x^3}{3} + 2 \frac{x^2}{2} + 4x + 8 \ln(x-2) + c$$

$$\Rightarrow \frac{x^3}{3} + x^2 + 4x + 8 \ln(x-2) + c \quad . \text{ (Where } c \text{ is some arbitrary constant)}$$

$$= \frac{x^3}{3} + x^2 + 4x + 8 \log |x-2| + c$$

$$3. \int \frac{x^2 + x + 5}{3x + 2} dx$$

Solution:

Given

$$\int \frac{x^2 + x + 5}{3x + 2} dx$$

By doing long division of the given equation we get

$$\text{Quotient} = \frac{x}{3} + \frac{1}{9}$$

$$\text{Remainder} = \frac{43}{9}$$

∴ We can write the above equation as

$$\Rightarrow \frac{x}{3} + \frac{1}{9} + \frac{43}{9} \left(\frac{1}{3x+2} \right)$$

∴ The above equation becomes

$$\Rightarrow \int \frac{x}{3} + \frac{1}{9} + \frac{43}{9} \left(\frac{1}{3x+2} \right) dx$$

$$\Rightarrow \frac{1}{3} \int x dx + \frac{1}{9} \int dx + \frac{43}{9} \int \frac{1}{3x+2} dx$$

$$\text{We know } \int x dx = \frac{x^n}{n+1}; \int \frac{1}{x} dx = \ln x$$

$$\Rightarrow \frac{1}{3} \times \frac{x^2}{2} + \frac{1}{9} \times x + \frac{43}{9} \ln(3x+2) + c$$

$$\Rightarrow \frac{x^2}{6} + \frac{x}{9} + \frac{43}{9} \ln(3x+2) + c. \text{ (Where } c \text{ is some arbitrary constant)}$$

$$= \frac{x^2}{6} + \frac{1}{9}x + \frac{43}{27} \log |3x+2| + c$$

∴ The above equation becomes

$$\Rightarrow \int \frac{x}{3} + \frac{1}{9} + \frac{43}{9} \left(\frac{1}{3x+2} \right) dx$$

$$\Rightarrow \frac{1}{3} \int x dx + \frac{1}{9} \int dx + \frac{43}{9} \int \frac{1}{3x+2} dx$$

We know $\int x^n dx = \frac{x^{n+1}}{n+1}; \int \frac{1}{x} dx = \ln x$

$$\Rightarrow \frac{1}{3} \times \frac{x^2}{2} + \frac{1}{9} \times x + \frac{43}{9} \times \frac{1}{3} \ln(3x+2) + c$$

$$= \frac{x^2}{6} + \frac{1}{9}x + \frac{43}{27} \log |3x+2| + c \text{ (Where } c \text{ is some arbitrary constant)}$$

Exercise 19.5 Page No: 19.33

1. $\int \frac{x+1}{\sqrt{2x+3}} dx$

Solution:

Given

$$\int \frac{x+1}{\sqrt{2x+3}} dx$$

In this type of questions, little manipulation makes the questions easier to solve

Here we have multiply and divide by 2 to given equation

$$\Rightarrow \frac{1}{2} \int \frac{2x+2}{\sqrt{2x+3}} dx$$

Add and subtract 1 from the numerator

$$\Rightarrow \frac{1}{2} \int \frac{2x+2+1-1}{\sqrt{2x+3}} dx$$

$$\Rightarrow \frac{1}{2} \int \frac{2x+3-1}{\sqrt{2x+3}} dx$$

Splitting the above equation we get

$$\Rightarrow \frac{1}{2} \int \frac{2x+3}{\sqrt{2x+3}} dx - \frac{1}{2} \int \frac{1}{\sqrt{2x+3}} dx$$

Taking $\frac{1}{2}$ common from the above equation

$$\Rightarrow \frac{1}{2} \left(\int \sqrt{2x+3} dx - \int (2x+3)^{-\frac{1}{2}} dx \right)$$

Now by integrating the above equation we get

$$\Rightarrow \frac{1}{2} \times \frac{(2x+3)^{\frac{3}{2}}}{2 \times \frac{3}{2}} - \frac{1}{2} \times \frac{(2x+3)^{\frac{1}{2}}}{2 \times \frac{1}{2}} + c$$

$$\Rightarrow \frac{(2x+3)^{\frac{3}{2}}}{6} - \frac{(2x+3)^{\frac{1}{2}}}{2} + c$$

2. $\int x\sqrt{x+2} dx$

Solution:

Given

$$\int x\sqrt{x+2} dx$$

In this type of questions, little manipulation makes the questions easier to solve

Here add and subtract 2 from x in the given equation

We get

$$\Rightarrow \int (x + 2 - 2)\sqrt{x + 2} dx$$

$$\Rightarrow \int (x + 2)^{\frac{3}{2}} dx - \int 2\sqrt{x + 2} dx$$

On integrating we get

$$\Rightarrow \frac{2(x+2)^{\frac{5}{2}}}{5} - \frac{4(x+2)^{\frac{3}{2}}}{3} + c$$

3. $\int \frac{x-1}{\sqrt{x+4}} dx$

Solution:

Given

$$\int \frac{x-1}{\sqrt{x+4}} dx$$

In this type of questions, little manipulation makes the questions easier to solve

Add and subtract 5 from the numerator

$$\Rightarrow \int \frac{x+5-5-1}{\sqrt{x+4}} dx$$

$$\Rightarrow \int \frac{x+4-5}{\sqrt{x+4}} dx$$

$$\Rightarrow \int \frac{x+4}{\sqrt{x+4}} dx - \int \frac{5}{\sqrt{x+4}} dx$$

$$\Rightarrow \left(\int \sqrt{x+4} dx - 5 \int (x+4)^{-\frac{1}{2}} dx \right)$$

$$\Rightarrow \int \frac{x+4-5}{\sqrt{x+4}} dx$$

By splitting the above equation

$$\Rightarrow \int \frac{x+4}{\sqrt{x+4}} dx - \int \frac{5}{\sqrt{x+4}} dx$$

$$\Rightarrow \left(\int \sqrt{x+4} dx - 5 \int (x+4)^{-\frac{1}{2}} dx \right)$$

Now by integrating, we get

$$\Rightarrow \frac{(x+4)^{\frac{3}{2}}}{\frac{3}{2}} - 5 \times \frac{(x+4)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

By computing

$$\Rightarrow \frac{2(x+4)^{\frac{3}{2}}}{3} - 10(x+4)^{\frac{1}{2}} + c$$

4. $\int (x+2)\sqrt{3x+5} dx$

Solution:

Let

$$I = \int (x + 2) \sqrt{3x + 5} dx$$

Substitute $3x + 5 = t$

$$\Rightarrow x = \frac{t - 5}{3}$$

$$\Rightarrow 3dx = dt$$

$$\Rightarrow dx = \frac{dt}{3}$$

$$\begin{aligned} \therefore I &= \int \left(\frac{t - 5}{3} + 2 \right) \sqrt{t} \frac{dt}{3} \\ &= \frac{1}{3} \int \left(\frac{t - 5 + 6}{3} \right) \sqrt{t} dt \end{aligned}$$

By taking 3 as common and multiplying, we get

$$= \frac{1}{9} \int \left(t^{\frac{3}{2}} + t^{\frac{1}{2}} \right) dt$$

On integrating we get

By taking 3 as common and multiplying, we get

$$= \frac{1}{9} \int \left(t^{\frac{3}{2}} + t^{\frac{1}{2}} \right) dt$$

On integrating we get

$$= \frac{1}{9} \left[\frac{t^{\frac{3}{2} + 1}}{\frac{3}{2} + 1} + \frac{t^{\frac{1}{2} + 1}}{\frac{1}{2} + 1} \right] + C$$

By taking 3 as common and multiplying, we get

$$= \frac{1}{9} \int \left(t^{\frac{3}{2}} + t^{\frac{1}{2}} \right) dt$$

On integrating we get

$$= \frac{1}{9} \left[\frac{t^{\frac{3}{2} + 1}}{\frac{3}{2} + 1} + \frac{t^{\frac{1}{2} + 1}}{\frac{1}{2} + 1} \right] + C$$

On simplifying

$$= \frac{1}{9} \left[\frac{2}{5} t^{\frac{5}{2}} + \frac{2}{3} t^{\frac{3}{2}} \right] + C$$

By substituting the value of t

$$\begin{aligned} &= \frac{1}{9} \left[\frac{2}{5} (3x + 5)^{\frac{5}{2}} + \frac{2}{3} (3x + 5)^{\frac{3}{2}} \right] + C \\ &= \frac{2}{9} \left[(3x + 5)^{\frac{3}{2}} \left\{ \frac{3x + 5}{5} + \frac{1}{3} \right\} \right] + C \\ &= \frac{2}{9} \left[(3x + 5)^{\frac{3}{2}} \left\{ \frac{9x + 15 + 5}{15} \right\} \right] + C \\ &= \frac{2}{9} \left[(3x + 5)^{\frac{3}{2}} \left\{ \frac{9x + 20}{15} \right\} \right] + C \\ &= \frac{2}{135} (3x + 5)^{\frac{3}{2}} (9x + 20) + C \end{aligned}$$

5. $\int \frac{2x + 1}{\sqrt{3x + 2}} dx$

Solution:

Given

$$\int \left(\frac{2x + 1}{\sqrt{3x + 2}} \right) dx$$

Multiply and divide by 3 in the above equation we get

$$= \frac{1}{3} \int \left(\frac{6x + 3}{\sqrt{3x + 2}} \right) dx$$

The above equation can be written as

$$= \frac{1}{3} \int \left(\frac{6x + 4 - 1}{\sqrt{3x + 2}} \right) dx$$

Taking 2 as common and subtracting

The above equation can be written as

$$= \frac{1}{3} \int \left(\frac{6x+4-1}{\sqrt{3x+2}} \right) dx$$

Taking 2 as common and subtracting

$$= \frac{1}{3} \int \left(\frac{2(3x+2)}{\sqrt{3x+2}} - \frac{1}{\sqrt{3x+2}} \right) dx$$

On simplifying

$$= \frac{1}{3} \int \left(2\sqrt{3x+2} - \frac{1}{\sqrt{3x+2}} \right) dx$$

By splitting the integral

$$= \frac{1}{3} \left[\int 2(3x+2)^{\frac{1}{2}} dx - \int (3x+2)^{-\frac{1}{2}} dx \right]$$

On integrating we get

$$\begin{aligned} &= \frac{1}{3} \left[2 \left\{ \frac{(3x+2)^{\frac{1}{2}+1}}{3(\frac{1}{2}+1)} \right\} - \frac{(3x+2)^{-\frac{1}{2}+1}}{(-\frac{1}{2}+1) \times 3} \right] + C \\ &= \frac{1}{3} \left[\frac{4}{9} (3x+2)^{\frac{3}{2}} - \frac{2}{3} (3x+2)^{\frac{1}{2}} \right] + C \end{aligned}$$

On simplifying we get

$$\begin{aligned} &= \frac{4}{27} (3x+2)^{\frac{3}{2}} - \frac{2}{9} (3x+2)^{\frac{1}{2}} + C \\ &= \sqrt{3x+2} \left(\frac{4}{27} (3x+2) - \frac{2}{9} \right) + C \\ &= \sqrt{3x+2} \left(\frac{4(3x+2)-6}{27} \right) + C \\ &= \sqrt{3x+2} \left(\frac{12x+8-6}{27} \right) + C \\ &= \frac{2}{27} (6x+1) \sqrt{3x+2} + C \end{aligned}$$

Exercise 19.6 Page No: 19.36

1. $\int \sin^2(2x+5) dx$

Solution:

We know that

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

By substituting the above formula

∴ The given equation becomes,

$$\Rightarrow \int \frac{1 - \cos 2(2x+5)}{2} dx$$

$$\text{We know } \int \cos ax dx = \frac{1}{a} \sin ax + c$$

$$\Rightarrow \frac{1}{2} \int dx - \frac{1}{2} \int \cos(4x + 10) dx$$

On integrating we get

$$\Rightarrow \frac{x}{2} - \frac{1}{8} \sin(4x + 10) + c$$

$$2. \int \sin^3(2x + 1) dx$$

Solution:

$$\text{We know that } \sin 3x = -4\sin^3 x + 3\sin x$$

The above formula can be written as

$$\Rightarrow 4\sin^3 x = 3\sin x - \sin 3x$$

The above equation becomes

$$\Rightarrow \sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

Now applying above formula to the given question we get

$$\Rightarrow \int \sin^3(2x + 1) dx = \int \frac{3 \sin(2x+1) - \sin 3(2x+1)}{4} dx$$

We know $\int \sin ax dx = \frac{-1}{a} \cos ax + c$

By substituting the above formula we get

$$\Rightarrow \frac{3}{8} \int \sin(2x + 1) dx - \frac{1}{4} \int \sin(6x + 3) dx$$

On integrating we get

$$\Rightarrow \frac{-3}{8} \cos(2x + 1) + \frac{1}{24} \cos(6x + 3) + c$$

3. $\int \cos^4 2x dx$

Solution:

Consider,

$$\cos^4 2x = (\cos^2 2x)^2$$

We know that

$$\Rightarrow \cos^2 x = \frac{1 + \cos 2x}{2}$$

The above equation

$$\Rightarrow (\cos^2 2x)^2 = \left(\frac{1 + \cos 4x}{2} \right)^2$$

$$\Rightarrow \left(\frac{1 + \cos 4x}{2} \right)^2 = \left(\frac{1 + 2\cos 4x + \cos^2 4x}{4} \right)$$

$$\Rightarrow \cos^2 4x = \frac{1 + \cos 8x}{2}$$

$$\Rightarrow \left(\frac{1 + 2\cos 4x + \cos^2 4x}{4} \right) = \frac{1}{4} + \frac{\cos 4x}{2} + \frac{1 + \cos 8x}{8}$$

Now the question becomes,

$$\Rightarrow \frac{1}{4} \int dx + \frac{1}{2} \int \cos 4x dx + \frac{1}{8} \int dx + \frac{1}{8} \int \cos 8x dx$$

$$\text{We know } \int \cos ax dx = \frac{1}{a} \sin ax + c$$

$$\Rightarrow \frac{1}{4} \int dx + \frac{1}{2} \int \cos 4x dx + \frac{1}{8} \int dx + \frac{1}{8} \int \cos 8x dx$$

$$\text{We know } \int \cos ax dx = \frac{1}{a} \sin ax + c$$

$$\Rightarrow \frac{x}{4} + \frac{1}{8} \sin 4x + \frac{x}{8} + \frac{\sin 8x}{64} + c$$

$$\Rightarrow \frac{24x + 8\sin 4x + \sin 8x}{64} + c$$

$$4. \int \sin^2 bx dx$$

Solution:

We know that

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

By substituting this formula,

∴ The given equation becomes,

$$\Rightarrow \int \frac{1 - \cos 2bx}{2} dx$$

$$\text{We know } \int \cos ax dx = \frac{1}{a} \sin ax + c$$

$$\Rightarrow \frac{1}{2} \int dx - \frac{1}{2} \int \cos(2bx) dx$$

On integration

$$\Rightarrow \frac{x}{2} - \frac{1}{4b} \sin(2bx) + c$$

Exercise 19.7 Page No: 19.38

Integrate the following integrals:

1. $\int \sin 4x \cos 7x dx$

Solution:

Given

$$\int \sin 4x \cos 7x \, dx$$

We know that $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$

Now by substituting this formula in given question we get

$$\therefore \sin 4x \cos 7x = \frac{\sin 11x + \sin(-3x)}{2}$$

We know $\sin(-\theta) = -\sin \theta$

Hence $\sin(-3x) = -\sin 3x$

$\therefore$ the above equation becomes

$$\Rightarrow \int \frac{1}{2} (\sin 11x - \sin 3x) \, dx$$

$$\Rightarrow \frac{1}{2} (\int \sin 11x \, dx - \int \sin 3x \, dx)$$

We know $\int \sin ax \, dx = \frac{-1}{a} \cos ax + c$

$$\Rightarrow \frac{1}{2} \left(\frac{-1}{11} \cos 11x + \frac{1}{3} \cos 3x \right)$$

$$= -\frac{1}{22} \cos 11x + \frac{1}{6} \cos 3x + c$$

2. $\int \cos 3x \cos 4x \, dx$

Solution:

Given

$$\int \cos 4x \cos 3x \, dx$$

Multiply and divide the given equation by 2

$$= \frac{1}{2} \int 2 \cos 4x \cos 3x \, dx$$

We know that $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$

$$= \frac{1}{2} \int [\cos (4x + 3x) + \cos (4x - 3x)] \, dx$$

Now by simplifying we get

$$= \frac{1}{2} \int (\cos (7x) + \cos x) \, dx$$

On integration we get

$$= \frac{1}{2} \left[\frac{\sin 7x}{7} + \sin x \right] + C$$

$$= \frac{1}{14} \sin 7x + \frac{1}{2} \sin x + C$$

$$3. \int \cos mx \cos nx \, dx, m \neq n$$

Solution:

Given

$$\int \cos mx \cos nx \, dx, m \neq n$$

We know $2 \cos A \cos B = \cos (A - B) + \cos (A + B)$

Now substituting the above formula we get,

$$\therefore \cos mx \cos nx = \frac{\cos(m-n)x + \cos(m+n)x}{2}$$

∴ The above equation becomes

$$\Rightarrow \int \frac{1}{2}(\cos(m-n)x + \cos(m+n)x)dx$$

We know $\int \cos ax dx = \frac{1}{a} \sin ax + c$

Applying the above

$$\Rightarrow \frac{1}{2} \left(\frac{1}{m-n} \sin(m-n)x + \frac{1}{m+n} \sin(m+n)x \right)$$

$$\Rightarrow \frac{1}{2} \left(\frac{(m+n) \sin(m-n)x + (m-n) \sin(m+n)x}{m^2 - n^2} \right) + c$$

We know that $a^2 - b^2 = (a+b)(a-b)$

By substituting the above formula and simplifying we get

$$\frac{1}{2} \left\{ \frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right\} + c$$

Exercise 19.8 Page No: 19.47

Evaluate the following integrals:

1. $\int \frac{1}{\sqrt{1 - \cos 2x}} dx$

Solution:

Given

$$\int \frac{1}{\sqrt{1 - \cos 2x}} dx$$

In the given equation $\cos 2x = \cos^2 x - \sin^2 x$

Also we know $\cos^2 x + \sin^2 x = 1$.

Substituting the values in the above equation we get

$$\Rightarrow \int \frac{1}{\sqrt{\sin^2 x + \cos^2 x - (-\sin^2 x + \cos^2 x)}} dx$$

$$\Rightarrow \int \frac{1}{\sqrt{\sin^2 x + \cos^2 x + \sin^2 x - \cos^2 x}} dx$$

$$\Rightarrow \int \frac{1}{\sqrt{2\sin^2 x}} dx$$

$$\Rightarrow \int \frac{1}{\sqrt{2}\sin x} dx$$

$$\frac{1}{\sqrt{2}} \int \operatorname{cosec} x dx$$

$$\Rightarrow \frac{1}{\sqrt{2}} \int \csc x dx$$

$$\Rightarrow \frac{1}{\sqrt{2}} \log \left| \frac{\tan x}{2} \right| + c$$

$$3. \int \frac{\sqrt{1 + \cos 2x}}{\sqrt{1 - \cos 2x}} dx$$

Solution:

Given,

$$\int \frac{\sqrt{1 + \cos 2x}}{\sqrt{1 - \cos 2x}} dx$$

Given

$$\int \frac{\sqrt{1 + \cos 2x}}{\sqrt{1 - \cos x}} dx$$

We know that

$$1 + \cos 2x = 2 \cos^2 x$$

$$1 - \cos 2x = 2 \sin^2 x$$

By substituting these formulae in the given equation we get

$$\Rightarrow \int \sqrt{\frac{2 \cos^2 x}{2 \sin^2 x}} dx$$

Again by applying standard formula, we get

$$\Rightarrow \int \sqrt{\cot^2 x} dx$$

By simplifying we get

$$\Rightarrow \int \cot x dx$$

$$\Rightarrow \log |\sin x| + c$$

$$4. \int \frac{\sqrt{1 - \cos x}}{\sqrt{1 + \cos x}} dx$$

Solution:

Given,

$$\int \frac{\sqrt{1 - \cos x}}{\sqrt{1 + \cos x}} dx$$

Given

$$\int \frac{\sqrt{1 - \cos 2x}}{\sqrt{1 + \cos x}} dx$$

We know that

$$1 - \cos x = 2 \sin^2 \frac{x}{2}$$

$$1 + \cos x = 2 \cos^2 \frac{x}{2}$$

By substituting these formulae in the given equation we get

$$\Rightarrow \int \sqrt{\tan^2 \frac{x}{2}} dx$$

On simplification,

$$\Rightarrow \int \tan \frac{x}{2} dx$$

$$\Rightarrow -2 \ln \left| \cos \frac{x}{2} \right| + c$$

$$5. \int \frac{\sec x}{\sec 2x} dx$$

Solution:

Here first of all convert sec x in terms of cos x

We know

$$\Rightarrow \sec x = \frac{1}{\cos x}, \sec 2x = \frac{1}{\cos 2x}$$

Therefore the above equation becomes,

$$\Rightarrow \frac{\frac{1}{\cos x}}{\frac{1}{\cos 2x}}$$

$$= \frac{\cos 2x}{\cos x}$$

∴ The equation now becomes

$$\Rightarrow \int \frac{\cos 2x}{\cos x} dx$$

We know

$$\cos 2x = 2 \cos^2 x - 1$$

∴ We can write the above equation as

$$\Rightarrow \int \frac{2 \cos^2 x - 1}{\cos x} dx$$

$$\Rightarrow \int 2 \cos x dx - \int \frac{1}{\cos x} dx$$

$$\Rightarrow 2 \sin x - \int \sec x dx$$

$$(\int \sec x dx = \ln|\sec x + \tan x| + c)$$

$$\Rightarrow 2 \sin x - \int \sec x dx$$

$$(\int \sec x dx = \ln|\sec x + \tan x| + c)$$

$$\Rightarrow 2 \sin x - \log |\sec x + \tan x| + c$$

$$6. \int \frac{\cos 2x}{(\cos x + \sin x)^2} dx$$

Solution:

Let

$$I = \int \frac{\cos 2x}{(\cos x + \sin x)^2} dx$$

By substituting the formula, we get

$$= \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2} dx$$

On simplification, we get

$$= \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$$

Put $\sin x + \cos x = t$

$$\Rightarrow -\sin x + \cos x = \frac{dt}{dx}$$

On rearranging

$$\Rightarrow (\cos x - \sin x) dx = dt$$

$$\therefore I = \int \frac{1}{t} dt$$

$$= \ln |t| + C$$

Now substitute the value of t , we get

$$= \ln |\cos x + \sin x| + C$$

$$7. \int \frac{\sin(x-a)}{\sin(x-b)}$$

Solution:

To solve these types of questions, it is better to eliminate the denominator.

$$\Rightarrow \int \frac{\sin(x-a)}{\sin(x-b)} dx$$

Add and subtract b in (x - a)

$$\Rightarrow \int \frac{\sin(x-a+b-b)}{\sin(x-b)} dx$$

$$\Rightarrow \int \frac{\sin(x-b+b-a)}{\sin(x-b)} dx$$

Numerator is of the form $\sin(A+B) = \sin A \cos B + \cos A \sin B$

Where $A = x - b$; $B = b - a$

$$\Rightarrow \int \frac{\sin(x-b) \cos(b-a) + \cos(x-b) \sin(b-a)}{\sin(x-b)} dx$$

$$\Rightarrow \int \frac{\sin(x-b) \cos(b-a)}{\sin(x-b)} dx + \int \frac{\cos(x-b) \sin(b-a)}{\sin(x-b)} dx$$

$$\Rightarrow \int \cos(b-a) dx + \int \cot(x-b) \sin(b-a) dx$$

$$\Rightarrow \cos(b-a) \int dx + \sin(b-a) \int \cot(x-b) dx$$

$$\text{As } \int \cot(x) dx = \ln |\sin x|$$

$$\Rightarrow \cos(b-a)x + \sin(b-a) \log |\sin(x-b)|$$

Therefore,

$= \cos(b-a)x + \sin(b-a) \log |\sin(x-b)| + c$, where c is an arbitrary constant.

Exercise 19.9 Page No: 19.57

Evaluate the following integrals:

$$1. \int \frac{\log x}{x} dx$$

Solution:

Assume $\log x = t$

$$\Rightarrow d(\log x) = dt$$

$$\Rightarrow \frac{1}{x} dx = dt$$

Substituting t and dt in above equation we get

$$\Rightarrow \int t \cdot dt$$

$$\Rightarrow \frac{t^2}{2} + c$$

But $t = \log(x)$

$$\Rightarrow \frac{\log^2 x}{2} + c$$

$$2. \int \frac{\log \left(1 + \frac{1}{x}\right)}{x(1+x)} dx$$

Solution:

$$\text{Assume } \log \left(1 + \frac{1}{x}\right) = t$$

$$\Rightarrow d\left(\log \left(1 + \frac{1}{x}\right)\right) = dt$$

$$\Rightarrow \frac{1}{1 + \frac{1}{x}} \times \frac{-1}{x^2} dx = dt$$

$$\Rightarrow \frac{x}{x+1} \times \frac{-1}{x^2} dx = dt$$

$$\Rightarrow \frac{-1 \cdot dx}{x(x+1)} = dt$$

$$\Rightarrow \frac{-1 \cdot dx}{x(x+1)} = dt$$

$$\Rightarrow \frac{dx}{x(x+1)} = -dt$$

∴ Substituting t and dt in the given equation we get

$$\Rightarrow \int -t \cdot dt$$

$$\Rightarrow -\int t \cdot dt$$

$$\Rightarrow \frac{-t^2}{2} + c$$

$$\text{But } \log\left(1 + \frac{1}{x}\right) = t$$

$$\Rightarrow -\frac{1}{2} \left\{ \log\left(1 + \frac{1}{x}\right) \right\}^2 + c$$

$$3. \int \frac{(1 + \sqrt{x})^2}{\sqrt{x}} dx$$

Solution:

Assume $1 + \sqrt{x} = t$

$$\Rightarrow d(1 + \sqrt{x}) = dt$$

$$\Rightarrow \frac{1}{2\sqrt{x}} dx = dt$$

$$\Rightarrow \frac{1}{\sqrt{x}} dx = 2dt$$

$\therefore$ Substituting t and dt in the given equation we get

$$\Rightarrow \int 2t^2 \cdot dt$$

$$\Rightarrow 2 \int t^2 \cdot dt$$

$$\Rightarrow \frac{2t^3}{3} + c$$

But $1 + \sqrt{x} = t$

$$\Rightarrow \frac{2(1 + \sqrt{x})^3}{3} + c.$$

4. $\int \sqrt{1 + e^x} e^x dx$

Solution:

Assume $1 + e^x = t$

$$\Rightarrow d(1 + e^x) = dt$$

$$\Rightarrow e^x dx = dt$$

$\therefore$ Substituting t and dt in given equation we get

$$\Rightarrow \int \sqrt{t} \cdot dt$$

$$\Rightarrow \int t^{1/2} \cdot dt$$

$$\Rightarrow \frac{2t^{3/2}}{3} + c$$

But $1 + e^x = t$

$$\Rightarrow \frac{2(1 + e^x)^{3/2}}{3} + c$$

5. $\int \sqrt[3]{\cos^2 x} \sin x \, dx$

Solution:

Assume $\cos x = t$

$$\Rightarrow d(\cos x) = dt$$

$$\Rightarrow -\sin x \, dx = dt$$

$$\Rightarrow dx = \frac{-dt}{\sin x}$$

$\therefore$ Substituting t and dt in the given equation we get

$$\Rightarrow \int \sqrt[3]{t^2} \sin x \cdot \frac{dt}{\sin x}$$

∴ Substituting t and dt in the given equation we get

$$\Rightarrow \int \sqrt[3]{t^2} \sin x \cdot \frac{-dt}{\sin x}$$

$$\Rightarrow \int -t^{2/3} \cdot dt$$

$$\Rightarrow -\frac{3}{5} t^{5/3} + c$$

But $\cos x = t$

$$\Rightarrow -\frac{3}{5} \cos^{5/3} x + c$$

$$6. \int \frac{e^x}{(1 + e^x)^2} dx$$

Solution:

Assume $1 + e^x = t$

$$\Rightarrow d(1 + e^x) = dt$$

$$\Rightarrow e^x dx = dt$$

∴ Substituting t and dt in given equation we get

$$\Rightarrow \int \frac{1}{t^2} dt$$

$$\Rightarrow \int t^{-2} \cdot dt$$

$$\Rightarrow \frac{-1}{t} + c$$

But $1 + e^x = t$

$$\Rightarrow \frac{-1}{1 + e^x} + c$$

$$7. \int \cot^3 x \operatorname{cosec}^2 x \, dx$$

Solution:

Assume $\cot x = t$

$$\Rightarrow d(\cot x) = dt$$

$$\Rightarrow -\operatorname{cosec}^2 x \cdot dx = dt$$

$$\Rightarrow dx = \frac{-dt}{\operatorname{csc}^2 x}$$

$\therefore$ Substituting t and dt in the given equation we get

$$\Rightarrow \int t^3 \operatorname{csc}^2 x \cdot \frac{-dt}{\operatorname{csc}^2 x}$$

$$\Rightarrow \int -t^3 \cdot dt$$

$$\Rightarrow -\int t^3 \cdot dt$$

$$\Rightarrow \frac{-t^4}{4} + C$$

But $t = \cot x$

$$\Rightarrow \frac{-\cot^4 x}{4} + C$$

$$8. \int \frac{\left\{e^{\sin^{-1} x}\right\}^2}{\sqrt{1-x^2}} \, dx$$

Solution:

Assume $\sin^{-1}x = t$

$$\Rightarrow d(\sin^{-1}x) = dt$$

$$\Rightarrow \frac{dx}{\sqrt{1-x^2}} = dt$$

$\therefore$ Substituting t and dt in the given equation we get

$$\Rightarrow \int e^{t^2} dt$$

$$\Rightarrow \int e^{2t} \cdot dt$$

$$\Rightarrow \frac{e^{2t}}{2} + c$$

But $t = \sin^{-1}x$

$$\Rightarrow \frac{1}{2} \{ e^{\sin^{-1}x} \}^2 + c$$

$$\Rightarrow \frac{e^{2t}}{2} + c$$

But $t = \sin^{-1}x$

$$\Rightarrow \frac{1}{2} \{ e^{\sin^{-1}x} \}^2 + c$$

$$9. \int \frac{1 + \sin x}{\sqrt{x - \cos x}} dx$$

Solution:

Assume $x - \cos x = t$

$$\Rightarrow d(x - \cos x) = dt$$

$$\Rightarrow (1 + \sin x) dx = dt$$

$\therefore$ Substituting t and dt in given equation we get

$$\Rightarrow \int \frac{1}{\sqrt{t}} dt$$

$$\Rightarrow \int t^{-1/2} \cdot dt$$

$$\Rightarrow 2t^{1/2} + c$$

But $t = x - \cos x$.

$$\Rightarrow 2(x - \cos x)^{1/2} + c.$$

$$10. \int \frac{1}{\sqrt{1-x^2}(\sin^{-1} x)^2} dx$$

Solution:

Assume $\sin^{-1} x = t$

$$\Rightarrow d(\sin^{-1} x) = dt$$

$$\Rightarrow \frac{dx}{\sqrt{1-x^2}} = dt$$

$\therefore$ Substituting t and dt in the given equation we get

$$\Rightarrow \int \frac{1}{t^2} dt$$

$$\Rightarrow \int t^{-2} \cdot dt$$

On integrating the above equation we get

$$\Rightarrow \frac{t^{-1}}{-1} + c$$

But $t = \sin^{-1} x$

$$\Rightarrow \frac{-1}{\sin^{-1} x} + c$$

Exercise 19.10 Page No: 19.65

1. $\int x^2 \sqrt{x+2} dx$

Solution:

$$\text{Let } I = \int x^2 \sqrt{x+2} dx$$

Substituting, $x+2 = t \Rightarrow dx = dt$,

$$I = \int (t-2)^2 \sqrt{t} dt$$

$$\Rightarrow I = \int (t^2 - 4t + 4) \sqrt{t} dt$$

$$\Rightarrow I = \int \left(t^{\frac{5}{2}} - 4t^{\frac{3}{2}} + 4t^{\frac{1}{2}} \right) dt$$

$$\Rightarrow I = \frac{2}{7} t^{\frac{7}{2}} - \frac{8}{5} t^{\frac{5}{2}} + \frac{8}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I = \frac{2}{7} (x+2)^{\frac{7}{2}} - \frac{8}{5} (x+2)^{\frac{5}{2}} + \frac{8}{3} (x+2)^{\frac{3}{2}} + c$$

$$\text{Therefore, } \int x^2 \sqrt{x+2} dx = \frac{2}{7} (x+2)^{\frac{7}{2}} - \frac{8}{5} (x+2)^{\frac{5}{2}} + \frac{8}{3} (x+2)^{\frac{3}{2}} + c$$

$$2. \int \frac{x^2}{\sqrt{x-1}} dx$$

Solution:

$$\text{Let } I = \int \frac{x^2}{\sqrt{x-1}} dx$$

Substituting $x-1 = t \Rightarrow dx = dt$,

Now substituting the values we get

$$\Rightarrow I = \int \frac{(t+1)^2}{\sqrt{t}} dt$$

Expanding using $(a+b)^2$ formula

$$\Rightarrow I = \int \frac{(t + 1)^2}{\sqrt{t}} dt$$

Expanding using $(a + b)^2$ formula

$$\Rightarrow I = \int \frac{t^2 + 2t + 1}{\sqrt{t}} dt$$

On simplification

$$\Rightarrow I = \int \left(t^{\frac{3}{2}} + 2t^{\frac{1}{2}} + t^{-\frac{1}{2}} \right) dt$$

On integrating we get

$$\Rightarrow I = \frac{2}{5} t^{\frac{5}{2}} + 2t^{\frac{1}{2}} + \frac{4}{3} t^{\frac{3}{2}} + c$$

Again taking LCM

$$\Rightarrow I = \frac{\left(6t^{\frac{5}{2}} + 30t^{\frac{1}{2}} + 20t^{\frac{3}{2}} \right)}{15} + c$$

$$\Rightarrow I = \frac{2}{15} t^{\frac{1}{2}} (3t^2 + 15 + 10t) + c$$

Substituting the value of t we get

$$\Rightarrow I = \frac{2}{15} (x - 1)^{\frac{1}{2}} (3(x - 1)^2 + 15 + 10(x - 1)) + c$$

$$\Rightarrow I = \frac{2}{15} (x - 1)^{\frac{1}{2}} (3(x^2 - 2x + 1)^2 + 15 + 10x - 10) + c$$

$$\Rightarrow I = \frac{2}{15} (x-1)^{\frac{1}{2}} (3(x^2 - 2x + 1) + 15 + 10x - 10) + c$$

By simplifying we get

$$\Rightarrow I = \frac{2}{15} (x-1)^{\frac{1}{2}} (3x^2 + 4x + 8) + c$$

$$\text{Therefore, } \int \frac{x^2}{\sqrt{x-1}} dx = \frac{2}{15} (x-1)^{\frac{1}{2}} (3x^2 + 4x + 8) + c$$

$$3. \int \frac{x^2}{\sqrt{3x+4}} dx$$

Solution:

$$\text{Let } I = \int \frac{x^2}{\sqrt{3x+4}} dx$$

Substituting $3x + 4 = t \Rightarrow 3 dx = dt$,

Substituting the values of x

$$\Rightarrow I = \int \frac{\left(\frac{t-4}{3}\right)^2}{3\sqrt{t}} dt$$

Expanding the above given function using $(a - b)^2$ formula

$$\Rightarrow I = \frac{1}{27} \int \frac{t^2 + 16 - 8t}{\sqrt{t}} dt$$

On simplifying, we get

$$\Rightarrow I = \frac{1}{27} \int \left(t^{\frac{3}{2}} - 8t^{\frac{1}{2}} + 16t^{-\frac{1}{2}} \right) dt$$

On integrating, we get

$$\Rightarrow I = \frac{1}{27} \left[\frac{2}{5} t^{\frac{5}{2}} - \frac{16}{3} t^{\frac{3}{2}} + 32t^{\frac{1}{2}} \right] + c$$

$$\Rightarrow I = \frac{1}{27} \left[\frac{2}{5} (3x + 4)^{\frac{5}{2}} - \frac{16}{3} (3x + 4)^{\frac{3}{2}} + 32(3x + 4)^{\frac{1}{2}} \right] + c$$

$$\Rightarrow I = \frac{2}{135} (3x + 4)^{\frac{5}{2}} - \frac{16}{81} (3x + 4)^{\frac{3}{2}} + \frac{32}{27} (3x + 4)^{\frac{1}{2}} + c$$

$$\begin{aligned} \text{Therefore, } \int \frac{x^2}{\sqrt{3x+4}} dx \\ = \frac{2}{135} (3x + 4)^{\frac{5}{2}} - \frac{16}{81} (3x + 4)^{\frac{3}{2}} + \frac{32}{27} (3x + 4)^{\frac{1}{2}} + c \end{aligned}$$

$$4. \int \frac{2x - 1}{(x - 1)^2} dx$$

Solution:

$$\text{Let } I = \int \frac{2x-1}{(x-1)^2} dx$$

Substituting $x - 1 = t \Rightarrow dx = dt$

Substituting the values of x

$$\Rightarrow I = \int \frac{2(t+1)-1}{t^2} dt$$

Multiplying and simplifying we get

$$\Rightarrow I = \int \frac{2t+1}{t^2} dt$$

$$\Rightarrow I = \int \left(\frac{2}{t} + \frac{1}{t^2} \right) dt$$

On integration

$$\Rightarrow I = 2 \log|t| + \frac{1}{t} + c$$

$$\Rightarrow I = 2 \log|x-1| + \frac{1}{x-1} + c$$

$$\text{Therefore, } \int \frac{2x-1}{(x-1)^2} dx = 2 \log|x-1| + \frac{1}{x-1} + c$$

On integration

$$\Rightarrow I = 2 \log|t| - \frac{1}{t} + c$$

$$\Rightarrow I = 2 \log|x-1| - \frac{1}{x-1} + c$$

$$\text{Therefore, } \int \frac{2x-1}{(x-1)^2} dx = 2 \log|x-1| - \frac{1}{x-1} + c$$

$$5. \int (2x^2 + 3)\sqrt{x+2} dx$$

Solution:

$$\text{Let } I = \int (2x^2 + 3)\sqrt{x+2} dx$$

Substituting $x + 2 = t \Rightarrow dx = dt$

Substituting the values of x in given equation, we get

$$\Rightarrow I = \int [2(t-2)^2 + 3]\sqrt{t} dt$$

$$\Rightarrow I = \int [2(t-2)^2 + 3]\sqrt{t} dt$$

Expanding above equation using $(a-b)^2$ formula

$$\Rightarrow I = \int [2t^2 - 8t + 8 + 3]\sqrt{t} dt$$

On simplification

$$\Rightarrow I = \int \left[2t^{\frac{5}{2}} - 8t^{\frac{3}{2}} + 11t^{\frac{1}{2}} \right] dt$$

On integrating we get

$$\Rightarrow I = \frac{4}{7} t^{\frac{7}{2}} - \frac{16}{5} t^{\frac{5}{2}} + \frac{22}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I = \frac{4}{7} (x+2)^{\frac{7}{2}} - \frac{16}{5} (x+2)^{\frac{5}{2}} + \frac{22}{3} (x+2)^{\frac{3}{2}} + c$$

$$\therefore \int (2x^2 + 3)\sqrt{x+2} dx = \frac{4}{7} (x+2)^{\frac{7}{2}} - \frac{16}{5} (x+2)^{\frac{5}{2}} + \frac{22}{3} (x+2)^{\frac{3}{2}} + c$$

Exercise 19.11 Page No: 19.69

Evaluate the following integrals:

$$1. \int \tan^3 x \sec^2 x dx$$

Solution:

$$\text{Let } I = \int \tan^3 x \sec^2 x \, dx$$

Let $\tan x = t$, then

$$\Rightarrow \sec^2 x \, dx = dt$$

Substituting the values of x

$$\Rightarrow I = \int t^3 \, dt$$

On integrating we get

$$\Rightarrow I = \frac{t^4}{4} + c$$

Substituting the value of t we get

$$\Rightarrow I = \frac{\tan^4 x}{4} + c$$

$$\text{Therefore, } \int \tan^3 x \sec^2 x \, dx = \frac{\tan^4 x}{4} + c$$

$$2. \int \tan x \sec^4 x \, dx$$

Solution:

$$\text{Let } I = \int \tan x \sec^4 x \, dx$$

The above equation can be written as

$$\Rightarrow I = \int \tan x \sec^2 x \sec^2 x \, dx$$

$$\Rightarrow I = \int \tan x \sec^2 x \sec^2 x \, dx$$

$$\Rightarrow I = \int \tan x (1 + \tan^2 x) \sec^2 x \, dx$$

$$\Rightarrow I = \int (\tan x + \tan^3 x) \sec^2 x \, dx$$

Let $\tan x = t$, then

$$\Rightarrow \sec^2 x \, dx = dt$$

Substituting the values of x

$$\Rightarrow I = \int (t + t^3) dt$$

On integrating we get

$$\Rightarrow I = \frac{t^2}{2} + \frac{t^4}{4} + c$$

$$\Rightarrow I = \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + c$$

$$\text{Therefore, } \int \tan x \sec^4 x \, dx = \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + c$$

$$3. \int \tan^5 x \sec^4 x \, dx$$

Solution:

$$\text{Let } I = \int \tan^5 x \sec^4 x \, dx$$

The above equation can be written as

$$\Rightarrow I = \int \tan^5 x \sec^2 x \sec^2 x \, dx$$

Taking $\tan^5 x$ as common

$$\Rightarrow I = \int \tan^5 x (1 + \tan^2 x) \sec^2 x \, dx$$

$$\Rightarrow I = \int \tan^5 x (1 + \tan^2 x) \sec^2 x \, dx$$

On simplifying

$$\Rightarrow I = \int (\tan^5 x + \tan^7 x) \sec^2 x \, dx$$

Let $\tan x = t$, then

$$\Rightarrow \sec^2 x \, dx = dt$$

Substituting the value of x

$$\Rightarrow I = \int (t^5 + t^7) dt$$

Integrating we get

$$\Rightarrow I = \frac{t^6}{6} + \frac{t^8}{8} + c$$

Substituting the values of t

$$\Rightarrow I = \frac{\tan^6 x}{6} + \frac{\tan^8 x}{8} + c$$

$$\text{Therefore, } \int \tan^5 x \sec^4 x \, dx = \frac{\tan^6 x}{6} + \frac{\tan^8 x}{8} + c$$

$$4. \int \sec^6 x \tan x \, dx$$

Solution:

$$\text{Let } I = \int \sec^6 x \tan x \, dx$$

The above equation can be written as

$$\Rightarrow I = \int \sec^5 x (\sec x \tan x) \, dx$$

Substituting, $\sec x = t \Rightarrow \sec x \tan x \, dx = dt$

$$\Rightarrow I = \int t^5 dt$$

On integrating we get

$$\Rightarrow I = \frac{t^6}{6} + c$$

Now substituting the values of t we get

$$\Rightarrow I = \frac{\sec^6 x}{6} + c$$

$$\text{Therefore, } \int \sec^5 x (\sec x \tan x) dx = \frac{\sec^6 x}{6} + c$$

$$5. \int \tan^5 x \, dx$$

Solution:

$$\text{Let } I = \int \tan^5 x \, dx$$

The above equation can be written as

$$\Rightarrow I = \int \tan^2 x \tan^3 x \, dx$$

Using standard formula

$$\Rightarrow I = \int (\sec^2 x - 1) \tan^3 x \, dx$$

Splitting the above equation we get

$$\Rightarrow I = \int \tan^3 x \sec^2 x \, dx - \int \tan^3 x \, dx$$

$$\Rightarrow I = \int \tan^3 x \sec^2 x \, dx - \int (\sec^2 x - 1) \tan x \, dx$$

$$\Rightarrow I = \int \tan^3 x \sec^2 x \, dx - \int (\sec^2 x \tan x) \, dx + \int \tan x \, dx$$

Let $\tan x = t$, then

$$\Rightarrow \sec^2 x \, dx = dt$$

$$\Rightarrow I = \int t^3 \, dt - \int t \, dt + \int \tan x \, dx$$

$$\Rightarrow I = \frac{t^4}{4} - \frac{t^2}{2} + \log|\sec x| + c$$

Let $\tan x = t$, then

$$\Rightarrow \sec^2 x \, dx = dt$$

$$\Rightarrow I = \int t^3 dt - \int t dt + \int \tan x \, dx$$

$$\Rightarrow I = \frac{t^4}{4} - \frac{t^2}{2} + \log|\sec x| + c$$

$$\Rightarrow I = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log|\sec x| + c$$

$$\text{Therefore, } \int \tan^5 x \, dx = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log|\sec x| + c$$

$$6. \int \sqrt{\tan x} \sec^4 x \, dx$$

Solution:

$$\text{Let } I = \int \sqrt{\tan x} \sec^4 x \, dx$$

The above equation can be written as

$$\Rightarrow I = \int \sqrt{\tan x} \sec^2 x \sec^2 x \, dx$$

Taking common

$$\Rightarrow I = \int \sqrt{\tan x} (1 + \tan^2 x) \sec^2 x \, dx$$

$$\Rightarrow I = \int (\tan^{\frac{1}{2}} x + \tan^{\frac{5}{2}} x) \sec^2 x \, dx$$

Let $\tan x = t$, then

$$\Rightarrow \sec^2 x \, dx = dt$$

$$\Rightarrow I = \int \left(t^{\frac{1}{2}} + t^{\frac{5}{2}} \right) dt$$

On integrating we get

$$\Rightarrow I = \frac{2}{3} t^{\frac{3}{2}} + \frac{2}{7} t^{\frac{7}{2}} + c$$

Substituting the value of t

On integrating we get

$$\Rightarrow I = \frac{2}{3} t^{\frac{3}{2}} + \frac{2}{7} t^{\frac{7}{2}} + c$$

Substituting the value of t

$$\Rightarrow I = \frac{2}{3} \tan^{\frac{3}{2}} x + \frac{2}{7} \tan^{\frac{7}{2}} x + c$$

$$\text{Therefore, } \int \sqrt{\tan x} \sec^4 x \, dx = \frac{2}{3} \tan^{\frac{3}{2}} x + \frac{2}{7} \tan^{\frac{7}{2}} x + c$$

$$1. \int \sin^4 x \cos^3 x \, dx$$

Solution:

Let

$$\sin x = t$$

We know the Differentiation of $\sin x = \cos x$

$$dt = d(\sin x) = \cos x \, dx$$

$$\text{So, } dx = \frac{dt}{\cos x}$$

Substitute all in above equation,

$$\int \sin^4 x \cos^3 x \, dx = \int t^4 \cos^3 x \frac{dt}{\cos x}$$

$$= \int t^4 \cos^2 x \, dt$$

$$= \int t^4 (1 - \sin^2 x) \, dt$$

$$= \int t^4 (1 - t^2) \, dt$$

$$= \int (t^4 - t^6) \, dt$$

We know, basic integration formula, $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ for any $c \neq -1$

$$\text{Hence, } \int (t^4 - t^6) \, dt = \frac{t^5}{5} - \frac{t^7}{7} + c$$

Put back $t = \sin x$

$$\int \sin^4 x \cos^3 x \, dx = \frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + c$$

$$2. \int \sin^5 x \, dx$$

Solution:

The given equation can be written as

$$\begin{aligned}\int \sin^5 x \, dx &= \int \sin^3 x \sin^2 x \, dx \\&= \int \sin^3 x (1 - \cos^2 x) \, dx \quad \{\text{since } \sin^2 x + \cos^2 x = 1\} \\&= \int (\sin^3 x - \sin^3 x \cos^2 x) \, dx \\&= \int (\sin x (\sin^2 x) - \sin^3 x \cos^2 x) \, dx \\&= \int (\sin x (1 - \cos^2 x) - \sin^3 x \cos^2 x) \, dx \quad \{\text{since } \sin^2 x + \cos^2 x = 1\} \\&= \int (\sin x - \sin x \cos^2 x - \sin^3 x \cos^2 x) \, dx \\&= \int \sin x \, dx - \int \sin x \cos^2 x \, dx - \int \sin^3 x \cos^2 x \, dx \quad (\text{separate the integrals})\end{aligned}$$

We know, $d(\cos x) = -\sin x \, dx$

So put $\cos x = t$ and $dt = -\sin x \, dx$ in above integrals

$$\begin{aligned}&= \int \sin x \, dx - \int \sin x \cos^2 x \, dx - \int \sin^3 x \cos^2 x \, dx \\&= \int \sin x \, dx - \int t^2 (-dt) - \int (\sin^2 x \sin x) t^2 \, dx \\&= \int \sin x \, dx - \int t^2 (-dt) - \int (1 - \cos^2 x) t^2 (-dt) \\&= \int \sin x \, dx + \int t^2 dt + \int (1 - t^2) t^2 dt \\&= \int \sin x \, dx + \int t^2 dt + \int (t^2 - t^4) dt \\&= -\cos x + \frac{t^3}{3} + \frac{t^3}{3} - \frac{t^5}{5} + c \quad (\text{since } \int x^n \, dx = \frac{x^{n+1}}{n+1} + c \text{ for any } c \neq -1)\end{aligned}$$

Put back $t = \cos x$

$$\begin{aligned}&= -\cos x + \frac{t^3}{3} + \frac{t^3}{3} - \frac{t^5}{5} + c \\&= -\cos x + \frac{\cos^3 x}{3} + \frac{\cos^3 x}{3} - \frac{\cos^5 x}{5} + c \\&= -\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + c = -\left[\cos x - \frac{2}{3} \cos^3 x + \frac{1}{5} \cos^5 x\right] + c\end{aligned}$$

$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c = -\left[\cos x - \frac{2}{3}\cos^3 x + \frac{1}{5}\cos^5 x\right] + c$$

$$3. \int \cos^5 x \, dx$$

Solution:

The given question can be written as

$$\int \cos^5 x \, dx = \int \cos^3 x \cos^2 x \, dx$$

$$= \int \cos^3 x (1 - \sin^2 x) \, dx \quad \{\text{since } \sin^2 x + \cos^2 x = 1\}$$

$$= \int (\cos^3 x - \cos^3 x \sin^2 x) \, dx$$

$$= \int (\cos x (\cos^2 x) - \cos^3 x \sin^2 x) \, dx$$

$$= (\cos x (1 - \sin^2 x) - \cos^3 x \sin^2 x) \, dx \quad \{\text{since } \sin^2 x + \cos^2 x = 1\}$$

$$= \int (\cos x - \cos x \sin^2 x - \cos^3 x \sin^2 x) \, dx$$

$$= \int \cos x \, dx - \int \cos x \sin^2 x \, dx - \int \cos^3 x \sin^2 x \, dx \quad (\text{separate the integrals})$$

We know, $d(\sin x) = \cos x \, dx$

So put $\sin x = t$ and $dt = \cos x \, dx$ in above integrals

$$= \int \cos x \, dx - \int t^2 \, dt - \int \cos x \cos^2 x \sin^2 x \, dx$$

$$= \int \cos x \, dx - \int t^2 \, (dt) - \int (\cos^2 x \cos x) t^2 \, dx$$

$$= \int \cos x \, dx - \int t^2 \, (dt) - \int (1 - \sin^2 x) t^2 \, (dt)$$

$$= \int \cos x \, dx - \int t^2 \, dt - \int (1 - t^2) t^2 \, dt$$

$$= \int \cos x \, dx - \int t^2 \, dt - \int (t^2 - t^4) \, dt$$

$$= \sin x - \frac{t^3}{3} - \frac{t^3}{3} + \frac{t^5}{5} + c$$

Put back $t = \sin x$

$$= \sin x - \frac{\sin^3 x}{3} - \frac{\sin^3 x}{3} + \frac{\cos^5 x}{5} + c$$

$$= \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + c$$

4. $\int \sin^5 x \cos x \, dx$

Solution:

Let $\sin x = t$

Then $d(\sin x) = dt = \cos x \, dx$

Put $t = \sin x$ and $dt = \cos x \, dx$ in given equation

$$\int \sin^5 x \cos x \, dx = \int t^5 dt$$

On integrating we get

$$= \frac{t^6}{6} + c$$

Substituting the value of t

$$= \frac{\sin^6 x}{6} + c$$

5. $\int \sin^3 x \cos^6 x \, dx$

Solution:

Since power of $\sin$ is odd, put $\cos x = t$

Then $dt = -\sin x \, dx$

Substitute these in above equation,

$$\int \sin^3 x \cos^6 x \, dx = \int \sin x \sin^2 x t^6 \, dx$$

$$= \int (1 - \cos^2 x) t^6 \sin x \, dx$$

$$= \int (1 - t^2) t^6 dt$$

$$= \int -(1 - t^2) t^6 dt$$

$$= \int -(t^6 - t^8) dt$$

On integrating we get

$$= -\frac{t^7}{7} + \frac{t^9}{9} + c$$

Put the value of t we get

$$= -\frac{1}{7} \cos^7 x + \frac{1}{9} \cos^9 x + c$$

Exercise 19.13 Page No: 19.79

$$1. \int \frac{x^2}{(a^2 - x^2)^{\frac{3}{2}}} dx$$

Solution:

Given

$$\int \frac{x^2}{(a^2 - x^2)^{3/2}} dx$$

Put $x = a \sin \theta$, so $dx = a \cos \theta d\theta$ and $\theta = \sin^{-1}(x/a)$

Above equation becomes,

$$= \int \frac{a^2 \sin^2 \theta}{(a^2 - a^2 \sin^2 \theta)^{3/2}} (a \cos \theta d\theta) = \int \frac{a^2 \sin^2 \theta}{(a^2)(a^2 - a^2 \sin^2 \theta)^{3/2}} (a \cos \theta d\theta)$$

By taking a^2 common we get

$$\begin{aligned} &= \int \frac{a^2 \sin^2 \theta}{(a^2)^{3/2} (a^2 - a^2 \sin^2 \theta)^{3/2}} (a \cos \theta d\theta) = \int \sin^2 \theta * \frac{\cos \theta}{\cos^3 \theta} d\theta \\ &= \int \frac{\sin^2 \theta}{\cos^2 \theta} d\theta = \int \tan^2 \theta d\theta = \int (\sec^2 \theta - 1) d\theta \quad (\sec^2 \theta - 1 = \tan^2 \theta) \\ &= \int \sec^2 \theta d\theta - \int 1 d\theta = \tan \theta + c - \theta \\ &= \tan \theta - \theta + c \end{aligned}$$

Put $\theta = \sin^{-1}(x/a)$

$$\begin{aligned} &= \frac{x}{(\sqrt{a^2 - x^2})} - \sin^{-1} \frac{x}{a} + c \\ &= \int \sec^2 \theta d\theta - \int 1 d\theta \\ &= \tan \theta - \theta + c \end{aligned}$$

Put $\theta = \sin^{-1}(x/a)$

$$= \frac{x}{(\sqrt{a^2 - x^2})} - \sin^{-1} \frac{x}{a} + c$$

$$2. \int \frac{x^7}{(a^2 - x^2)^5} dx$$

Solution:

$$\text{Let } I = \int \frac{x^7}{(a^2 - x^2)^5} dx$$

$$\text{Let } x = a \sin \theta$$

On differentiating both sides we get

$$dx = a \cos \theta d\theta$$

$$\therefore I = \int \frac{a^8 \sin^7 \theta \cos \theta}{(a^2 - a^2 \sin^2 \theta)^5} d\theta$$

$$= \int \frac{a^8 \sin^7 \theta \cos \theta}{a^{10} (1 - \sin^2 \theta)^5} d\theta$$

$$= \int \frac{\sin^7 \theta}{a^2 \cos^9 \theta} d\theta$$

$$= \frac{1}{a^2} \int \tan^7 \theta \sec^2 \theta d\theta$$

Let

$$\tan \theta = t$$

Differentiating on both sides

$$\sec^2 \theta d\theta = dt$$

$$\therefore I = \frac{1}{a^2} \int t^7 dt$$

$$= \frac{1}{a^2} \frac{t^8}{8} + c$$

$$\begin{aligned}
&= \frac{1}{8a^2} (\tan^8 \theta) + c \\
&= \frac{1}{8a^2} \left(\tan \left(\sin^{-1} \frac{x}{a} \right) \right)^8 + c \\
&= \frac{1}{8a^2} \left(\tan \left(\tan^{-1} \frac{x}{\sqrt{a^2 - x^2}} \right) \right)^8 + c \\
&= \frac{1}{8a^2} \left(\frac{x}{\sqrt{a^2 - x^2}} \right)^8 + c \\
&= \frac{1}{8a^2} \frac{x^8}{(a^2 - x^2)^4} + c \\
&= \frac{1}{8a^2} \frac{x^8}{(a^2 - x^2)^4} + c
\end{aligned}$$

$$\text{Hence, } \int \frac{x^7}{(a^2 - x^2)^5} dx = \frac{1}{8a^2} \frac{x^8}{(a^2 - x^2)^4} + c$$

Exercise 19.14 Page No: 19.83

Evaluate the following integrals:

$$1. \int \frac{1}{a^2 - b^2 x^2} dx$$

Solution:

Taking out b^2 as common from the given equation, we get

$$\frac{1}{b^2} \int \frac{1}{\left(\frac{a^2}{b^2}\right) - x^2} dx$$

$$= \frac{1}{b^2} \int \frac{1}{\left(\frac{a^2}{b^2}\right) - x^2} dx = \frac{1}{b^2} \int \frac{1}{\left(\frac{a}{b}\right)^2 - x^2} dx$$

On integrating above equation using

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{x+a}{a-x} \right| + c \}, \text{ we get}$$

$$= \frac{1}{b^2} \times \frac{1}{2\left(\frac{a}{b}\right)} \log \left| \frac{\frac{a}{b} + x}{\frac{a}{b} - x} \right| + c$$

On simplification we get

$$= \frac{1}{2ab} \log \left| \frac{a + bx}{a - bx} \right| + c$$

$$2. \int \frac{1}{a^2 x^2 - b^2} dx$$

Solution:

Taking out a^2 as common from the given equation, we get

$$= \frac{1}{a^2} \int \frac{1}{x^2 - \frac{b^2}{a^2}} dx$$

On integrating above equation using

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \frac{x+a}{x-a} + c \} \text{ we get}$$

$$= \frac{1}{a^2} \int \frac{1}{x^2 - \left(\frac{b}{a}\right)^2} dx = \frac{1}{a^2} * \frac{1}{2\left(\frac{b}{a}\right)} \log \left[\frac{x - \left(\frac{b}{a}\right)}{x + \frac{b}{a}} \right] + c$$

On simplification

$$= \frac{1}{2ab} \log \frac{ax-b}{ax+b} + c$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$$

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \frac{x+a}{x-a} + c \} \text{ we get}$$

$$= \frac{1}{a^2} \int \frac{1}{x^2 - \left(\frac{b}{a}\right)^2} dx = \frac{1}{a^2} * \frac{1}{2\left(\frac{b}{a}\right)} \log \left[\frac{x - \left(\frac{b}{a}\right)}{x + \frac{b}{a}} \right] + c$$

On simplification

$$= \frac{1}{2ab} \log \frac{ax-b}{ax+b} + c$$

$$3. \int \frac{1}{a^2 x^2 + b^2} dx$$

Solution:

Taking out a^2 as common from the given equation, we get

$$= \frac{1}{a^2} \int \frac{1}{x^2 + \frac{b^2}{a^2}} dx$$

On integrating above equation using

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c, \text{ we get}$$

$$= \frac{1}{a^2} \int \frac{1}{x^2 + \left(\frac{b}{a}\right)^2} dx = \frac{1}{a^2} * \frac{1}{\left(\frac{b}{a}\right)} \tan^{-1} \left[\frac{x}{\frac{b}{a}} \right] + c$$

By simplifying we get

$$= \frac{1}{ab} \tan^{-1} \left(\frac{ax}{b} \right) + c$$

$$4. \int \frac{x^2 - 1}{x^2 + 4} dx$$

Solution:

Add and subtract 4 in the numerator of given equation, we get

$$= \int \frac{x^2 + 4 - 4 - 1}{x^2 + 4} dx = \int \frac{(x^2 + 4) - 4 - 1}{x^2 + 4} dx$$

Now separate the numerator terms, we get

$$= \int \frac{(x^2 + 4) - 5}{x^2 + 4} dx = \int \frac{(x^2 + 4)}{x^2 + 4} dx - \int \frac{5}{x^2 + 4} dx$$

On computing we get

$$= \int dx - \int \frac{5}{x^2 + 4} dx = \int dx - 5 \int \frac{1}{x^2 + 4} dx$$

$$\text{We know } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$$

$$= \int dx - 5 \int \frac{1}{x^2 + 2^2} dx = x - 5 \times \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + c$$

On integrating we get

$$= x - \frac{5}{2} \tan^{-1} \left(\frac{x}{2} \right) + c$$

Now separate the numerator terms, we get

$$= \int \frac{(x^2 + 4)^{-5}}{x^2 + 4} dx = \int \frac{(x^2 + 4)}{x^2 + 4} dx - \int \frac{5}{x^2 + 4} dx$$

On computing we get

$$= \int dx - \int \frac{5}{x^2 + 4} dx = \int dx - 5 \int \frac{1}{x^2 + 4} dx$$

We know $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$

$$= \int dx - 5 \int \frac{1}{x^2 + 2^2} dx = x - 5 \times \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + c$$

On integrating we get

$$= x - \frac{5}{2} \tan^{-1} \left(\frac{x}{2} \right) + c$$

5. $\int \frac{1}{\sqrt{1 + 4x^2}} dx$

Solution:

$$\text{Let } I = \int \frac{1}{\sqrt{1+4x^2}} dx$$

The above equation can be written as

$$= \int \frac{1}{\sqrt{1+(2x)^2}} dx$$

Let $t = 2x$, then $dt = 2dx$ or $dx = dt/2$

Therefore,

$$\int \frac{1}{\sqrt{1+(2x)^2}} dx = \frac{1}{2} \int \frac{dt}{\sqrt{1+t^2}}$$

$$\text{We know } \int \frac{1}{\sqrt{a^2+x^2}} dx = \log|x + \sqrt{a^2+x^2}| + c$$

$$= \frac{1}{2} \log|t + \sqrt{1+t^2}| + c$$

$$= \frac{1}{2} \log|2x + \sqrt{1+4x^2}| + c$$

Exercise 19.15 Page No: 19.86

$$1. \int \frac{1}{4x^2 + 12x + 5} dx$$

Solution:

Let

$$I = \int \frac{1}{4x^2 + 12x + 5} dx$$

Taking out $\frac{1}{4}$ as common, then we get

$$= \frac{1}{4} \int \frac{1}{x^2 + 3x + \frac{5}{4}} dx$$

Adding and subtracting $(\frac{3}{2})^2$ to the denominator

$$= \frac{1}{4} \int \frac{1}{x^2 + 2x \times \frac{3}{2} + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + \frac{5}{4}} dx$$

The above equation can be written as

$$= \frac{1}{4} \int \frac{1}{\left(x + \frac{3}{2}\right)^2 - 1} dx$$

Let

$$\left(x + \frac{3}{2}\right) = t \quad \dots (i)$$

$$\Rightarrow dx = dt$$

So, substituting the t values we get

$$I = \frac{1}{4} \int \frac{1}{t^2 - (1)^2} dt$$

$$I = \frac{1}{4} \times \frac{1}{2 \times 1} \log \left| \frac{t-1}{t+1} \right| + c$$

$$[\text{since, } \int \frac{1}{x^2 - (a)^2} dx = \frac{1}{2 \times a} \log \left| \frac{x-a}{x+a} \right| + c]$$

$$[\text{since, } \int \frac{1}{x^2 - (a)^2} dx = \frac{1}{2 \times a} \log \left| \frac{x-a}{x+a} \right| + c]$$

$$I = \frac{1}{8} \log \left| \frac{x + \frac{3}{2} - 1}{x + \frac{3}{2} + 1} \right| + c \quad [\text{Using (i)}]$$

$$I = \frac{1}{8} \log \left| \frac{2x + 1}{2x + 5} \right| + c$$

$$2. \int \frac{1}{x^2 - 10x + 34} dx$$

Solution:

Let

$$I = \int \frac{1}{x^2 - 10x + 34} dx$$

$$I = \int \frac{1}{x^2 - 10x + 34} dx$$

Adding and subtracting 5^2 to both sides

$$= \int \frac{1}{x^2 + 2x \times 5 + (5)^2 - (5)^2 + 34} dx$$

The above equation can be written as

$$= \int \frac{1}{(x - 5)^2 - 9} dx$$

$$\text{Let } (x - 5) = t \dots (i)$$

$$\Rightarrow dx = dt$$

So, substituting the values of t we get

$$I = \int \frac{1}{t^2 + (3)^2} dt$$

$$I = \frac{1}{3} \tan^{-1}\left(\frac{t}{3}\right) + c$$

$$[\text{since, } \int \frac{1}{x^2 + (a)^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c]$$

Adding and subtracting 5^2 to both sides

$$= \int \frac{1}{x^2 - 2x \times 5 + (5)^2 - (5)^2 + 34} dx$$

The above equation can be written as

$$= \int \frac{1}{(x-5)^2 + 9} dx$$

$$\text{Let } (x-5) = t \dots (i)$$

$$\Rightarrow dx = dt$$

So, substituting the values of t we get

$$I = \int \frac{1}{t^2 + (3)^2} dt$$

$$I = \frac{1}{3} \tan^{-1}\left(\frac{t}{3}\right) + c$$

$$[\text{since, } \int \frac{1}{x^2 + (a)^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c]$$

$$I = \frac{1}{3} \tan^{-1}\left(\frac{x-5}{3}\right) + c \quad [\text{Using (i)}]$$

$$I = \frac{1}{3} \tan^{-1}\left(\frac{x-5}{3}\right) + c$$

$$3. \int \frac{1}{1+x-x^2} dx$$

Solution:

$$\text{Let } I = \int \frac{1}{1+x-x^2} dx = \int \frac{1}{-(x^2-x-1)} dx$$

The above equation can be written as

$$= \int \frac{1}{-(x^2 - x - 1)} dx$$

Add and subtract $\frac{1}{4}$ to both sides

$$= \int \frac{1}{-(x^2 - x - \frac{1}{4} - 1 + \frac{1}{4})} dx$$

The above equation can be written as

$$= \int \frac{1}{-\left(\left(x - \frac{1}{2}\right)^2 - \frac{5}{4}\right)} dx$$

On computing we get

$$= \int \frac{1}{\left(\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2\right)} dx$$

$$I = \frac{1}{2 \times \frac{\sqrt{5}}{2}} \log \left| \frac{\frac{\sqrt{5}}{2} + \left(x - \frac{1}{2}\right)}{\frac{\sqrt{5}}{2} - \left(x - \frac{1}{2}\right)} \right| + c$$

$$[\text{since, } \int \frac{1}{x^2 - (a)^2} dx = \frac{1}{2 \times a} \log \left| \frac{x-a}{x+a} \right| + c]$$

By using,

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$$

$$[\text{since, } \int \frac{1}{x^2 - (a)^2} dx = \frac{1}{2 \times a} \log \left| \frac{x-a}{x+a} \right| + c]$$

$$I = \frac{1}{\sqrt{5}} \log \left| \frac{\sqrt{5} + 2x - 1}{\sqrt{5} - 2x + 1} \right| + c$$

$$I = \frac{1}{\sqrt{5}} \log \left| \frac{\sqrt{5} - 1 + 2x}{\sqrt{5} + 1 - 2x} \right| + c$$

$$4. \int \frac{1}{2x^2 - x - 1} dx$$

Solution:

$$\text{Let } I = \int \frac{1}{2x^2 - x - 1} dx$$

Taking out $\frac{1}{2}$ as common we get

$$= \frac{1}{2} \int \frac{1}{x^2 - \frac{x}{2} - \frac{1}{2}} dx$$

Again adding and subtracting $(\frac{1}{4})^2$ to the denominator we get

$$= \frac{1}{2} \int \frac{1}{x^2 + 2x \times \frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - \frac{1}{2}} dx$$

The above equation can be written as

$$= \frac{1}{2} \int \frac{1}{\left(x - \frac{1}{4}\right)^2 - \frac{9}{16}} dx$$

$$\text{Let } \left(x - \frac{1}{4}\right) = t \text{ (i)}$$

$$\Rightarrow dx = dt$$

$$I = \frac{1}{2} \int \frac{1}{t^2 - \left(\frac{3}{4}\right)^2} dt$$

So,

$$I = \frac{1}{2} \times \frac{1}{2 \times \frac{3}{4}} \log \left| \frac{t - \frac{3}{4}}{t + \frac{3}{4}} \right| + c$$

$$I = \frac{1}{2} \times \frac{1}{2 \times \frac{3}{4}} \log \left| \frac{t - \frac{3}{4}}{t + \frac{3}{4}} \right| + c$$

$$[\text{since, } \int \frac{1}{x^2 - (a)^2} dx = \frac{1}{2 \times a} \log \left| \frac{x - a}{x + a} \right| + c]$$

$$I = \frac{1}{3} \log \left| \frac{x - \frac{1}{4}}{x + \frac{1}{4}} \right| + c \quad [\text{Using (i)}]$$

$$I = \frac{1}{3} \log \left| \frac{2x - 2}{2x + 1} \right| + c$$

$$5. \int \frac{1}{x^2 + 6x + 13} dx$$

Solution:

In the denominator we have, and it can be written as

$$x^2 + 6x + 13 = x^2 + 6x + 3^2 - 3^2 + 13$$

The above equation can be written as

$$= (x + 3)^2 + 4$$

Substituting these values we get

$$\text{So, } \int \frac{1}{x^2 + 6x + 13} dx = \int \frac{1}{(x+3)^2 + 2^2} dx$$

Let $x+3 = t$

Then $dx = dt$

$$\int \frac{1}{(t)^2 + 2^2} dt = \frac{1}{2} \tan^{-1} \frac{t}{2} + c$$

$$[\text{since, } \int \frac{1}{x^2 + (a)^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c]$$

$$\frac{1}{2} \tan^{-1} \frac{x+3}{2} + c$$

Exercise 19.16 Page No: 19.90

Evaluate the following integrals:

$$1. \int \frac{\sec^2 x}{1 - \tan^2 x} dx$$

Solution:

$$\text{Let } I = \int \frac{\sec^2 x}{1 - \tan^2 x} dx$$

$$\text{Let } \tan x = t \dots (i)$$

$$\Rightarrow \sec^2 x dx = dt$$

So, substituting these values in given equation we get

$$I = \int \frac{dt}{(1)^2 - t^2}$$

$$I = \frac{1}{2 \times 1} \log \left| \frac{1+t}{1-t} \right| + c \quad \left[\text{since, } \int \frac{1}{a^2 - (x)^2} dx = \frac{1}{2 \times a} \log \left| \frac{a+x}{a-x} \right| + c \right]$$

$$I = \frac{1}{2} \log \left| \frac{1+\tan x}{1-\tan x} \right| + c \quad [\text{Using (i)}]$$

$$2. \int \frac{e^x}{1 + e^{2x}} dx$$

Solution:

$$\text{Let } I = \int \frac{e^x}{1 + e^{2x}} dx$$

$$\text{Let } e^x = t \dots (i)$$

$$\Rightarrow e^x dx = dt$$

So, substituting these values in given equation we get

$$I = \int \frac{dt}{(1)^2 + t^2}$$

$$I = \tan^{-1} t + c$$

$$[\text{since, } \int \frac{1}{1 + (x)^2} dx = \tan^{-1} x + c]$$

$$I = \tan^{-1}(e^x) + c \quad [\text{Using (i)}]$$

$$I = \tan^{-1} t + c$$

$$[\text{since, } \int \frac{1}{1 + (x)^2} dx = \tan^{-1} x + c]$$

$$I = \tan^{-1}(e^x) + c \text{ [Using (i)]}$$

$$3. \int \frac{\cos x}{\sin^2 x + 4 \sin x + 5} dx$$

Solution:

$$\text{Let } I = \int \frac{\cos x}{\sin^2 x + 4 \sin x + 5} dx$$

$$\text{Let } \sin x = t \dots (i)$$

$$\Rightarrow \cos x dx = dt$$

$$\text{So, } I = \int \frac{dt}{t^2 + 4t + 5}$$

Adding and subtracting 2^2 to the denominator we get

$$= \int \frac{dt}{t^2 + (2t)(2) + 2^2 - 2^2 + 5}$$

Above equation can be written as

$$\int \frac{dt}{(t + 2)^2 + 1}$$

$$\text{Again, let } t + 2 = u \dots (ii)$$

$$\Rightarrow dt = du$$

$$I = \int \frac{du}{u^2 + 1}$$

$$= \tan^{-1} u + c$$

$$[\text{since, } \int \frac{1}{1 + (x)^2} dx = \tan^{-1} x + c]$$

$$= \tan^{-1}(\sin x + 2) + c \text{ [Using (i), (ii)]}$$

$$4. \int \frac{e^x}{e^{2x} + 5e^x + 6} dx$$

Solution:

$$\text{Let } I = \int \frac{e^x}{e^{2x} + 5e^x + 6} dx$$

$$\text{Let } e^x = t \dots (i)$$

$$\Rightarrow e^x dx = dt$$

$$= \int \frac{1}{t^2 + 5t + 6} dt$$

$$= \int \frac{1}{t^2 + 2t \times \frac{5}{2} + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 6} dt$$

$$= \int \frac{1}{\left(t + \frac{5}{2}\right)^2 - \frac{1}{4}} dt$$

$$\text{Let } t + \frac{5}{2} = u \dots (ii)$$

$$\Rightarrow dt = du$$

So, substituting these values we get

$$I = \int \frac{1}{u^2 - \left(\frac{1}{2}\right)^2} du$$

$$I = \frac{1}{2 \times \frac{1}{2}} \log \left| \frac{u - \frac{1}{2}}{u + \frac{1}{2}} \right| + c$$

$$\text{Let } t + \frac{5}{2} = u \text{ (ii)}$$

$$\Rightarrow dt = du$$

So, substituting these values we get

$$I = \int \frac{1}{u^2 - \left(\frac{1}{2}\right)^2} du$$

$$I = \frac{1}{2 \times \frac{1}{2}} \log \left| \frac{u - \frac{1}{2}}{u + \frac{1}{2}} \right| + c$$

$$[\text{since, } \int \frac{1}{x^2 - (a)^2} dx = \frac{1}{2 \times a} \log \left| \frac{x - a}{x + a} \right| + c]$$

$$I = \log \left| \frac{2u - 1}{2u + 1} \right| + c$$

$$I = \log \left| \frac{2(t + \frac{5}{2}) - 1}{2(t + \frac{5}{2}) + 1} \right| + c \text{ [Using (ii)]}$$

$$I = \log \left| \frac{e^x + 2}{e^x + 3} \right| + c \text{ [Using (i)]}$$

$$5. \int \frac{e^{3x}}{4e^{6x} - 9} dx$$

Solution:

$$\text{Let } I = \int \frac{e^{3x}}{4e^{6x}-9} dx$$

$$\text{Let } e^{3x} = t \dots (i)$$

$$\Rightarrow 3e^{3x} dx = dt$$

$$I = \frac{1}{3} \int \frac{1}{4t^2 - 9} dt$$

Taking (1/4) as common we get

$$= \frac{1}{12} \int \frac{1}{t^2 - \frac{9}{4}} dt$$

The above equation can be written as

$$I = \frac{1}{12} \int \frac{1}{t^2 - \left(\frac{3}{2}\right)^2} dt$$

$$I = \frac{1}{36} \log \left| \frac{t - \frac{3}{2}}{t + \frac{3}{2}} \right| + c$$

$$[\text{since, } \int \frac{1}{x^2 - (a)^2} dx = \frac{1}{2 \times a} \log \left| \frac{x-a}{x+a} \right| + c]$$

$$I = \log \left| \frac{2t-3}{2t+3} \right| + c$$

$$I = \frac{1}{36} \log \left| \frac{2t-3}{2t+3} \right| + c$$

$$I = \frac{1}{36} \log \left| \frac{2e^{3x}-3}{2e^{3x}+3} \right| + c \quad [\text{Using (i)}]$$

Exercise 19.17 Page No: 19.93

Evaluate the following integrals:

$$1. \int \frac{1}{\sqrt{2x - x^2}} dx$$

Solution:

$$\text{Let } I = \int \frac{1}{\sqrt{2x - x^2}} dx$$

The above equation can be written as

$$= \int \frac{1}{\sqrt{-(x^2 - 2x)}} dx$$

Now by adding and subtracting 1^2 to the denominator we get

$$= \int \frac{1}{\sqrt{-(x^2 - 2x(1) + 1^2 - 1^2)}} dx$$

On simplifying

$$= \int \frac{1}{\sqrt{-(x - 1)^2 + 1}} dx$$

The above equation becomes

$$= \int \frac{1}{\sqrt{1 - (x - 1)^2}} dx$$

Let $(x - 1) = t$ and $dx = dt$

$$\text{So, } I = \int \frac{1}{\sqrt{1 - t^2}} dt$$

$$= \sin^{-1} t + c \text{ [since } \int \frac{1}{\sqrt{1 - x^2}} dx = \sin^{-1} x + c \text{]}$$

$$I = \sin^{-1}(x - 1) + c$$

$$2. \int \frac{1}{\sqrt{8 + 3x - x^2}} dx$$

Solution:

The denominator of given question $8 + 3x - x^2$ by adding and subtracting $(9/4)$ can be written as

$$8 - \left(x^2 - 3x + \frac{9}{4} - \frac{9}{4} \right)$$

Therefore

$$8 - \left(x^2 - 3x + \frac{9}{4} - \frac{9}{4} \right)$$

The above equation can be written as

$$= \frac{41}{4} - \left(x - \frac{3}{2} \right)^2$$

Substituting these values in given question we get

$$\int \frac{1}{\sqrt{8 + 3x - x^2}} dx = \int \frac{1}{\sqrt{\frac{41}{4} - \left(x - \frac{3}{2} \right)^2}} dx$$

Let $x - 3/2 = t$

$dx = dt$

$$\int \frac{1}{\sqrt{\frac{41}{4} - \left(x - \frac{3}{2} \right)^2}} dx = \int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2} \right)^2 - t^2}} dt$$

$$= \sin^{-1} \left(\frac{t}{\frac{\sqrt{41}}{2}} \right) + c$$

$$[\text{since } \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c]$$

$$= \sin^{-1} \left(\frac{x - \frac{3}{2}}{\frac{\sqrt{41}}{2}} \right) + c$$

$$= \sin^{-1} \left(\frac{2x - 3}{\sqrt{41}} \right) + c$$

$$3. \int \frac{1}{\sqrt{5 - 4x - 2x^2}} dx$$

Solution:

$$\text{Let } I = \int \frac{1}{\sqrt{5-4x-2x^2}} dx$$

Now taking out 2 as common from the denominator we get

$$= \int \frac{1}{\sqrt{-2 \left[x^2 + 2x - \frac{5}{2} \right]}} dx$$

By adding and subtracting 1^2 to the denominator we get

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{- \left[x^2 + 2x + (1)^2 - (1)^2 - \frac{5}{2} \right]}} dx$$

By computing

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{- \left[(x+1)^2 - \frac{7}{2} \right]}} dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\frac{7}{2} - (x+1)^2}} dx$$

Let $(x+1) = t$

Differentiating both sides, we get, $dx = dt$

$$I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(\frac{7}{2}\right)^2 - t^2}} dt$$

So,

$$I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(\sqrt{\frac{7}{2}}\right)^2 - t^2}} dt$$

So,

$$= \frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{t}{\sqrt{\frac{7}{2}}} \right) + c$$

$$[\text{since } \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c]$$

$$I = \frac{1}{\sqrt{2}} \sin^{-1} \left(\sqrt{\frac{2}{7}} \times (x+1) \right) + c$$

$$4. \int \frac{1}{\sqrt{3x^2 + 5x + 7}} dx$$

Solution:

$$\text{Let } I = \int \frac{1}{\sqrt{3x^2 + 5x + 7}} dx$$

Taking $1/\sqrt{3}$ as common from the denominator we get

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{x^2 + \frac{5}{3}x + \frac{7}{3}}} dx$$

Now by adding and subtracting $(5/6)^2$ to the denominator complete perfect square, we get

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{x^2 + 2x \left(\frac{5}{6}\right) + \left(\frac{5}{6}\right)^2 - \left(\frac{5}{6}\right)^2 + \frac{7}{3}}} dx$$

The above equation can be written as

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{\left(x + \frac{5}{6}\right)^2 - \frac{59}{36}}} dx$$

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{\left(x + \frac{5}{6}\right)^2 + \frac{59}{36}}} dx$$

$$\text{let } \left(x + \frac{5}{6}\right) = t$$

$$dx = dt$$

$$I = \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{t^2 + \left(\frac{\sqrt{59}}{6}\right)^2}} dt$$

$$= \frac{1}{\sqrt{3}} \log \left| t + \sqrt{t^2 + \left(\frac{\sqrt{59}}{6}\right)^2} \right| + c \quad \left[\text{since } \int \frac{1}{\sqrt{x^2 + a^2}} dx = \log |x + \sqrt{x^2 + a^2}| \right]$$

On simplification we get

$$I = \frac{1}{\sqrt{3}} \log \left| x + \frac{5}{6} + \sqrt{\left(x + \frac{5}{6}\right)^2 + \left(\frac{\sqrt{59}}{6}\right)^2} \right| + c$$

$$I = \frac{1}{\sqrt{3}} \log \left| x + \frac{5}{6} + \sqrt{x^2 + \frac{5x}{3} + \frac{7}{3}} \right| + c$$

Evaluate the following integrals:

1. $\int \frac{x}{\sqrt{x^4 + a^4}} dx$

Solution:

The given equation can be written as

$$\int \frac{x}{\sqrt{x^4 + a^4}} dx = \int \frac{x}{\sqrt{(x^2)^2 + (a^2)^2}} dx$$

Let $x^2 = t$, so $2x dx = dt$

Or, $x dx = dt/2$

$$\text{Hence, } \int \frac{x}{\sqrt{(x^2)^2 + (a^2)^2}} dx = \int \frac{1}{\sqrt{t^2 + (a^2)^2}} \frac{dt}{2} = \frac{1}{2} \int \frac{1}{\sqrt{t^2 + (a^2)^2}} dt$$

$$\text{Since, } \int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log|x + \sqrt{(x^2 + a^2)}| + c$$

$$\text{Hence, } \frac{1}{2} \int \frac{1}{\sqrt{t^2 + (a^2)^2}} dt = \frac{1}{2} \log |t + \sqrt{t^2 + (a^2)^2}| + c$$

Put $t = x^2$

$$= \frac{1}{2} \log |x^2 + \sqrt{(x^2)^2 + (a^2)^2}| + c$$

$$= \frac{1}{2} \log |x^2 + \sqrt{x^4 + a^4}| + c$$

$$2. \int \frac{\sec^2 x}{\sqrt{4 + \tan^2 x}} dx$$

Solution:

Let $\tan x = t$

Then $dt = \sec^2 x dx$

$$\text{Therefore, } \int \frac{\sec^2 x}{\sqrt{4 + \tan^2 x}} dx = \int \frac{dt}{\sqrt{2^2 + t^2}}$$

$$\text{Since, } \int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log|x + \sqrt{(x^2 + a^2)}| + c$$

$$\text{Hence, } \int \frac{dt}{\sqrt{2^2 + t^2}} = \log|t + \sqrt{t^2 + 2^2}| + c$$

$$= \log|\tan x + \sqrt{\tan^2 x + 4}| + c$$

$$3. \int \frac{e^x}{\sqrt{16 - e^{2x}}} dx$$

Solution:

Let $e^x = t$

Then we have, $e^x dx = dt$

Substituting these values,

Therefore, $\int \frac{e^x}{\sqrt{16-e^{2x}}} dx = \int \frac{dt}{\sqrt{4^2-t^2}}$

Since we have, $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

Hence, $\int \frac{dt}{\sqrt{4^2-t^2}} = \sin^{-1}\left(\frac{e^x}{4}\right) + c$

4. $\int \frac{\cos x}{\sqrt{4 + \sin^2 x}} dx$

Solution:

Let $\sin x = t$

Let $\sin x = t$

Then $dt = \cos x dx$

Now substituting these values we get

Hence, $\int \frac{\cos x}{\sqrt{4+\sin^2 x}} dx = \int \frac{dt}{\sqrt{2^2+t^2}}$

Since we have, $\int \frac{1}{\sqrt{(x^2+a^2)}} dx = \log[x + \sqrt{(x^2 + a^2)}] + c$

Therefore, $\int \frac{dt}{\sqrt{2^2+t^2}} = \log[t + \sqrt{t^2 + 2^2}] + c$

$= \log[t + \sqrt{t^2 + 2^2}] + c = \log[\sin x + \sqrt{\sin^2 x + 4}] + c$

5. $\int \frac{\sin x}{\sqrt{4 \cos^2 x - 1}} dx$

Solution:

Let

$$2\cos x = t$$

$$\text{Then } dt = -2\sin x \, dx$$

$$\text{Or, } \sin x \, dx = -\frac{dt}{2}$$

Then substituting these values we get,

$$\text{Therefore, } \int \frac{\sin x}{\sqrt{4\cos^2 x - 1}} dx = \int -\frac{dt}{2\sqrt{t^2 - 1^2}}$$

$$\text{Since, } \int \frac{1}{\sqrt{x^2 - a^2}} dx = \log[x + \sqrt{x^2 - a^2}] + c$$

$$\text{Therefore, } \int -\frac{dt}{2\sqrt{t^2 - 1^2}} = -\frac{1}{2} \log[t + \sqrt{t^2 - 1}] + c$$

On integrating we get

$$= -\frac{1}{2} \log[2\cos x + \sqrt{4\cos^2 x - 1}] + c$$

$$6. \int \frac{x}{\sqrt{4 - x^4}} dx$$

Solution:

Let $x^2 = t$

$$2x \, dx = dt \text{ or } x \, dx = dt/2$$

Now substituting these values in the given equation we get

$$\text{Hence, } \int \frac{x}{\sqrt{4-x^4}} dx = \int \frac{dt}{2(\sqrt{2^2-t^2})}$$

$$\text{Since we have, } \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c$$

$$\text{So, } \int \frac{dt}{2(\sqrt{2^2-t^2})} = \frac{1}{2} \sin^{-1} \left(\frac{t}{2} \right) + c$$

Put $t = x^2$

$$= \frac{1}{2} \sin^{-1} \left(\frac{t}{2} \right) + c = \frac{1}{2} \sin^{-1} \left(\frac{x^2}{2} \right) + c$$

$$7. \int \frac{1}{x\sqrt{4-9(\log x)^2}} dx$$

Solution:

Let $3 \log x = t$

We have $d(\log x) = 1/x$

Hence, $d(3 \log x) = dt = 3/x \, dx$

Or $1/x \, dx = dt/3$

Hence, $\int \frac{1}{x\sqrt{4-9(\log x)^2}} dx = \int \frac{1}{3\sqrt{2^2-t^2}} dt$

Since we have, $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

Hence, $\int \frac{1}{3\sqrt{2^2-t^2}} dt = \frac{1}{3} \sin^{-1}\left(\frac{t}{2}\right) + c$

Put $t = 3 \log x$

$$= \frac{1}{3} \sin^{-1}\left(\frac{t}{2}\right) + c = \frac{1}{3} \sin^{-1}\left(\frac{3 \log x}{2}\right) + c$$

Hence, $\int \frac{1}{3\sqrt{2^2-t^2}} dt = \frac{1}{3} \sin^{-1}\left(\frac{t}{2}\right) + c$

Put $t = 3 \log x$

$$= \frac{1}{3} \sin^{-1}\left(\frac{t}{2}\right) + c = \frac{1}{3} \sin^{-1}\left(\frac{3 \log x}{2}\right) + c$$

8. $\int \frac{\sin 8x}{\sqrt{9 + \sin^4 4x}} dx$

Solution:

Let $t = \sin^2 4x$

$$dt = 2 \sin 4x \cos 4x \times 4 dx$$

We know $\sin 2x = 2 \sin x \cos x$

Therefore, $dt = 4 \sin 8x dx$

Or, $\sin 8x dx = dt/4$

$$\int \frac{\sin 8x}{\sqrt{9 + \sin^4 4x}} dx = \frac{1}{4} \int \frac{dt}{\sqrt{3^2 + t^2}}$$

Since we have, $\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log[x + \sqrt{(x^2 + a^2)}] + c$

$$= \frac{1}{4} \int \frac{dt}{\sqrt{3^2 + t^2}} = \frac{1}{4} \log[t + \sqrt{t^2 + 3^2}] + c$$

$$= \frac{1}{4} \log[\sin^2 4x + \sqrt{9 + \sin^4 4x}] + c$$

$$9. \int \frac{\cos 2x}{\sqrt{\sin^2 2x + 8}} dx$$

Solution:

Let $t = \sin 2x$

$$dt = 2 \cos 2x dx$$

$$\cos 2x dx = dt/2$$

$$\int \frac{\cos 2x}{\sqrt{\sin^2 2x + 8}} dx = \frac{1}{2} \int dt / \sqrt{t^2 + (2\sqrt{2})^2}$$

$$\text{Since we have, } \int \frac{1}{\sqrt{x^2 + a^2}} dx = \log[x + \sqrt{x^2 + a^2}] + c$$

$$= \frac{1}{2} \int dt / \sqrt{t^2 + (2\sqrt{2})^2} = \frac{1}{2} \log[t + \sqrt{t^2 + 8}] + c$$

$$= \frac{1}{2} \log[t + \sqrt{t^2 + 8}] + c = \frac{1}{2} \log[\sin 2x + \sqrt{\sin^2 2x + 8}] + c$$

Exercise 19.19 Page No: 19.104

Evaluate the following integrals:

$$1. \int \frac{x}{x^2 + 3x + 2} dx$$

Solution:

Let

$$I = \int \frac{x}{x^2+3x+2} dx$$

As we can see that there is a term of x in numerator and derivative of x^2 is also $2x$. So there is a chance that we can make substitution for $x^2 + 3x + 2$ and I can be reduced to a fundamental integration.

$$\text{As, } \frac{d}{dx}(x^2 + 3x + 2) = 2x + 3$$

$$\therefore \text{ Let, } x = A(2x + 3) + B$$

$$\Rightarrow x = 2Ax + 3A + B$$

On comparing both sides

$$\text{We have, } 2A = 1 \Rightarrow A = 1/2$$

$$3A + B = 0 \Rightarrow B = -3A = -3/2$$

Hence,

$$I = \int \frac{\frac{1}{2}(2x+3) - \frac{3}{2}}{x^2+3x+2} dx$$

$$\therefore I = \frac{1}{2} \int \frac{2x+3}{x^2+3x+2} dx - \frac{3}{2} \int \frac{1}{x^2+3x+2} dx$$

$$\text{Let, } I_1 = \frac{1}{2} \int \frac{2x+3}{x^2+3x+2} dx \text{ and } I_2 = \frac{3}{2} \int \frac{1}{x^2+3x+2} dx$$

Now, $I = I_1 - I_2$ equation 1

We will solve I_1 and I_2 individually.

$$\text{As, } I_1 = \frac{1}{2} \int \frac{2x+3}{x^2+3x+2} dx$$

$$\text{Let } u = x^2 + 3x + 2 \Rightarrow du = (2x + 3) dx$$

$$\therefore I_1 \text{ reduces to } \frac{1}{2} \int \frac{du}{u}$$

Hence,

$$I_1 = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \log|u| + C$$

On substituting value of u, we have:

$$I_1 = \frac{1}{2} \log|x^2 + 3x + 2| + C \dots \text{Equation 2}$$

As, $I_2 = \frac{3}{2} \int \frac{1}{x^2+3x+2} dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will use to solve the problem.

$$\text{i) } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \quad \text{ii) } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$$\therefore I_2 = \frac{3}{2} \int \frac{1}{x^2+3x+2} dx$$

$$\Rightarrow I_2 = \frac{3}{2} \int \frac{1}{\{x^2 + 2(\frac{3}{2})x + (\frac{3}{2})^2\} + 2 - (\frac{3}{2})^2} dx$$

Using: $a^2 + 2ab + b^2 = (a + b)^2$

We have:

$$I_2 = \frac{3}{2} \int \frac{1}{\left(x + \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2} dx$$

$$I_2 \text{ matches with } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

$$I_2 \text{ matches with } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

$$\therefore I_2 = \frac{3}{2} \left\{ \frac{1}{2(\frac{1}{2})} \log \left| \frac{\left(x + \frac{3}{2}\right) - \frac{1}{2}}{\left(x + \frac{3}{2}\right) + \frac{1}{2}} \right| + C \right\}$$

$$\Rightarrow I_2 = \frac{3}{2} \log \left| \frac{2x+3-1}{2x+3+1} \right| + C$$

$$\Rightarrow I_2 = \frac{3}{2} \log \left| \frac{2x+2}{2x+4} \right| + C = \frac{3}{2} \log \left| \frac{x+1}{x+2} \right| + C \dots \text{equation 3}$$

From equation 1:

$$I = I_1 - I_2$$

Using equation 2 and equation 3:

$$I = \frac{1}{2} \log |x^2 + 3x + 2| - \frac{3}{2} \log \left| \frac{x+1}{x+2} \right| + C$$

$$2. \int \frac{x+1}{x^2+x+3} dx$$

Solution:

$$I = \int \frac{x+1}{x^2+x+3} dx$$

As we can see that there is a term of x in numerator and derivative of $x^2 + x + 3$ is $2x + 1$. So there is a chance that we can make substitution for $x^2 + x + 3$ and the integral can be reduced to a fundamental integration.

$$\text{As, } \frac{d}{dx}(x^2 + x + 3) = 2x + 1$$

$$\therefore \text{ Let, } x = A(2x + 1) + B$$

$$\Rightarrow x = 2Ax + A + B$$

On comparing both sides

We have,

$$2A = 1 \Rightarrow A = 1/2$$

$$A + B = 0 \Rightarrow B = -A = -1/2$$

Hence,

$$I = \int \frac{\frac{1}{2}(2x+1) - \frac{1}{2}}{x^2+x+3} dx$$

$$\therefore I = \frac{1}{2} \int \frac{2x+1}{x^2+x+3} dx - \frac{1}{2} \int \frac{1}{x^2+x+3} dx$$

$$\text{Let, } I_1 = \frac{1}{2} \int \frac{2x+1}{x^2+x+3} dx \text{ and } I_2 = \frac{1}{2} \int \frac{1}{x^2+x+3} dx$$

Now, $I = I_1 - I_2$ Equation 1

We will solve I_1 and I_2 individually.

$$\text{As } I_1 = \frac{1}{2} \int \frac{2x+1}{x^2+x+3} dx$$

$$\text{Let } u = x^2 + x + 3 \Rightarrow du = (2x + 1) dx$$

$$\therefore I_1 \text{ reduces to } \frac{1}{2} \int \frac{du}{u}$$

Hence,

$$I_1 = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \log|u| + C$$

On substituting the value of u , we have:

$$I_1 = \frac{1}{2} \log |x^2 + x + 3| + C \text{ equation 2}$$

As, $I_2 = \frac{1}{2} \int \frac{1}{x^2+x+3} dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will help to solve the problem.

$$\text{i) } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \quad \text{ii) } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$$\therefore I_2 = \frac{1}{2} \int \frac{1}{x^2 + x + 3} dx$$

$$\Rightarrow I_2 = \frac{1}{2} \int \frac{1}{\{x^2 + 2(\frac{1}{2})x + (\frac{1}{2})^2\} + 3 - (\frac{1}{2})^2} dx$$

$$\text{Using } a^2 + 2ab + b^2 = (a + b)^2$$

We have

$$I_2 = \frac{1}{2} \int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{11}}{2}\right)^2} dx$$

$$I_2 \text{ matches with } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

$$\therefore I_2 = \frac{1}{2} \left\{ \frac{1}{\left(\frac{\sqrt{11}}{2}\right)} \tan^{-1} \left(\frac{x + \frac{1}{2}}{\frac{\sqrt{11}}{2}} \right) + C \right\}$$

$$\Rightarrow I_2 = \frac{1}{\sqrt{11}} \tan^{-1} \left(\frac{2x+1}{\sqrt{11}} \right) + C \quad \dots \text{equation 3}$$

From equation 1 we have

$$I = I_1 - I_2$$

Using equation 2 and equation 3:

$$I = \frac{1}{2} \log|x^2 + x + 3| + \frac{1}{\sqrt{11}} \tan^{-1} \left(\frac{2x+1}{\sqrt{11}} \right) + C$$

Using equation 2 and equation 3:

$$I = \frac{1}{2} \log|x^2 + x + 3| - \frac{1}{\sqrt{11}} \tan^{-1} \left(\frac{2x+1}{\sqrt{11}} \right) + C$$

$$3. \int \frac{x-3}{x^2 + 2x - 4} dx$$

Solution:

$$\text{Let } I = \int \frac{x-3}{x^2+2x-4} dx$$

As we can see that there is a term of x in numerator and derivative of x^2 is also $2x$. So there is a chance that we can make substitution for $x^2 + 2x - 4$ and I can be reduced to a fundamental integration.

As we can see that there is a term of x in numerator and derivative of x^2 is also $2x$. So there is a chance that we can make substitution for $x^2 + 2x - 4$ and I can be reduced to a fundamental integration.

$$\text{As, } \frac{d}{dx}(x^2 + 2x - 4) = 2x + 2$$

$$\therefore \text{ Let, } x - 3 = A(2x + 2) + B$$

$$\Rightarrow x - 3 = 2Ax + 2A + B$$

$$\text{On comparing both sides we have, } 2A = 1 \Rightarrow A = 1/2$$

$$2A + B = -3 \Rightarrow B = -3 - 2A = -4$$

$$\text{Hence, } I = \int \frac{\frac{1}{2}(2x+2) - 4}{x^2+2x-4} dx$$

$$\therefore I = \frac{1}{2} \int \frac{2x+2}{x^2+2x-4} dx - 4 \int \frac{1}{x^2+2x-4} dx$$

$$\text{Let, } I_1 = \frac{1}{2} \int \frac{2x+2}{x^2+2x-4} dx \text{ and } I_2 = \int \frac{1}{x^2+2x-4} dx$$

$$\text{Now, } I = I_1 - 4I_2 \dots \text{equation 1}$$

We will solve I_1 and I_2 individually.

$$\text{As, } I_1 = \frac{1}{2} \int \frac{2x+2}{x^2+2x-4} dx$$

$$\text{Let } u = x^2 + 2x - 4 \Rightarrow du = (2x + 2) dx$$

$$\therefore I_1 \text{ reduces to } \frac{1}{2} \int \frac{du}{u}$$

$$\text{Hence, } I_1 = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \log|u| + C$$

On substituting value of u, we have:

$$I_1 = \frac{1}{2} \log|x^2 + 2x - 4| + C \quad \dots \text{Equation 2}$$

As, $I_2 = \int \frac{1}{x^2 + 2x - 4} dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

$$\text{i) } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \quad \text{ii) } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$$\therefore I_2 = \int \frac{1}{x^2 + 2x - 4} dx \Rightarrow I_2 = \int \frac{1}{\{x^2 + 2(1)x + (1)^2\} - 4 - (1)^2} dx$$

$$\text{Using } a^2 + 2ab + b^2 = (a + b)^2$$

We have:

$$I_2 = \int \frac{1}{(x+1)^2 - (\sqrt{5})^2} dx$$

$$I_2 \text{ matches with } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

$$\therefore I_2 = \frac{1}{2\sqrt{5}} \log \left| \frac{x+1-\sqrt{5}}{x+1+\sqrt{5}} \right| + C \quad \dots \text{equation 3}$$

From equation 1 we have

$$I = I_1 - 4I_2$$

Using equation 2 and equation 3:

$$I = \frac{1}{2} \log|x^2 + 2x - 4| - 4 \left(\frac{1}{2\sqrt{5}} \log \left| \frac{x+1-\sqrt{5}}{x+1+\sqrt{5}} \right| \right) + C$$

$$I = \frac{1}{2} \log|x^2 + 2x - 4| - \frac{2}{\sqrt{5}} \log \left| \frac{x+1-\sqrt{5}}{x+1+\sqrt{5}} \right| + C$$

$$4. \int \frac{2x - 3}{x^2 + 6x + 13} dx$$

Solution:

$$\text{Let } I = \int \frac{2x-3}{x^2+6x+13} dx$$

As we can see that there is a term of x in numerator and derivative of x^2 is also $2x$. So there is a chance that we can make a substitution for $x^2 + 6x + 13$ and I can be reduced to a fundamental integration.

$$\text{As } \frac{d}{dx}(x^2 + 6x + 13) = 2x + 6$$

$$\therefore \text{ Let, } 2x - 3 = A(2x + 6) + B$$

$$\Rightarrow 2x - 3 = 2Ax + 6A + B$$

On comparing both sides

$$\text{We have, } 2A = 2 \Rightarrow A = 1$$

$$6A + B = -3 \Rightarrow B = -3 - 6A = -9$$

$$\text{Hence, } I = \int \frac{(2x+6)-9}{x^2+6x+13} dx$$

$$\therefore I = \int \frac{2x+6}{x^2+6x+13} dx - 9 \int \frac{1}{x^2+6x+13} dx$$

$$\text{Let, } I_1 = \int \frac{2x+6}{x^2+6x+13} dx \text{ and } I_2 = \int \frac{1}{x^2+6x+13} dx$$

$$\text{Now, } I = I_1 - 9I_2 \dots \text{Equation 1}$$

We will solve I_1 and I_2 individually.

$$\text{As, } I_1 = \int \frac{2x+6}{x^2+6x+13} dx$$

$$\text{Let } u = x^2 + 6x + 13 \Rightarrow du = (2x + 6) dx$$

$\therefore I_1$ reduces to $\int \frac{du}{u}$

Hence, $I_1 = \int \frac{du}{u} = \log|u| + C$

On substituting value of u , we have

$$I_1 = \log|x^2 + 6x + 13| + C \text{ equation 2}$$

As, $I_2 = \int \frac{1}{x^2+6x+13} dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

$$\text{i) } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \quad \text{ii) } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$$\therefore I_2 = \int \frac{1}{x^2+6x+13} dx$$

$$\Rightarrow I_2 = \int \frac{1}{\{x^2+2(3)x+(3)^2\}+13-(3)^2} dx$$

Using $a^2 + 2ab + b^2 = (a + b)^2$

$$\text{We have } I_2 = \int \frac{1}{(x+3)^2 + (2)^2} dx$$

$$I_2 \text{ matches with } \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

$$\therefore I_2 = \frac{1}{2} \tan^{-1} \left(\frac{x+3}{2} \right) + C \text{ ... equation 3}$$

From equation 1

$$I = I_1 - 9I_2$$

Using equation 2 and equation 3:

$$I = \log|x^2 + 6x + 13| - 9 \frac{1}{2} \tan^{-1} \left(\frac{x+3}{2} \right) + C$$

$$I = \log|x^2 + 6x + 13| - \frac{9}{2} \tan^{-1} \left(\frac{x+3}{2} \right) + C$$

$$5. \int \frac{x-1}{3x^2-4x+3} dx$$

Solution:

$$\text{Let } I = \int \frac{x-1}{3x^2-4x+3} dx$$

As we can see that there is a term of x in numerator and derivative of x^2 is also $2x$. So there is a chance that we can make substitution for $3x^2-4x+3$ and I can be reduced to a fundamental integration.

$$\text{As, } \frac{d}{dx}(3x^2-4x+3) = 6x-4$$

$$\therefore \text{ Let, } x-1 = A(6x-4) + B$$

$$\Rightarrow x-1 = 6Ax-4A+B$$

On comparing both sides

$$\text{We have, } 6A = 1 \Rightarrow A = 1/6$$

$$-4A + B = -1 \Rightarrow B = -1 + 4A = -2/6 = -1/3$$

$$\text{Hence, } I = \int \frac{\frac{1}{6}(6x-4) - \frac{1}{3}}{3x^2-4x+3} dx$$

$$\therefore I = \frac{1}{6} \int \frac{6x-4}{3x^2-4x+3} dx - \frac{1}{3} \int \frac{1}{3x^2-4x+3} dx$$

$$\text{Let, } I_1 = \frac{1}{6} \int \frac{6x-4}{3x^2-4x+3} dx \text{ and } I_2 = \frac{1}{3} \int \frac{1}{3x^2-4x+3} dx$$

$$\text{Now, } I = I_1 - I_2 \dots \text{equation 1}$$

We will solve I_1 and I_2 individually.

$$\text{As, } I_1 = \frac{1}{6} \int \frac{6x-4}{3x^2-4x+3} dx$$

Let $u = 3x^2 - 4x + 3 \Rightarrow du = (6x - 4) dx$

$\therefore I_1$ reduces to $\frac{1}{6} \int \frac{du}{u}$

Hence,

$$I_1 = \frac{1}{6} \int \frac{du}{u} = \frac{1}{6} \log|u| + C$$

On substituting value of u , we have:

$$I_1 = \frac{1}{6} \log|3x^2 - 4x + 3| + C \quad \dots \text{equation 2}$$

$$I_1 = \frac{1}{6} \log|3x^2 - 4x + 3| + C \quad \dots \text{equation 2}$$

As, $I_2 = \frac{1}{3} \int \frac{1}{3x^2 - 4x + 3} dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

$$\text{i) } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \quad \text{ii) } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in the denominator

$$\therefore I_2 = \frac{1}{9} \int \frac{1}{x^2 - \frac{4}{3}x + 1} dx \quad \{\text{on taking 3 common from denominator}\}$$

$$\Rightarrow I_2 = \frac{1}{9} \int \frac{1}{\{x^2 - 2(\frac{2}{3})x + (\frac{2}{3})^2\} + 1 - (\frac{2}{3})^2} dx$$

Using $a^2 + 2ab + b^2 = (a + b)^2$

$$\text{We have } I_2 = \frac{1}{9} \int \frac{1}{\left(x - \frac{2}{3}\right)^2 + \left(\frac{\sqrt{5}}{3}\right)^2} dx$$

$$I_2 \text{ matches with } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

$$\therefore I_2 = \frac{1}{9} \frac{1}{\sqrt{5}} \tan^{-1} \left(\frac{x-2}{\frac{\sqrt{5}}{3}} \right) + C$$

$$\therefore I_2 = \frac{3}{9\sqrt{5}} \tan^{-1} \left(\frac{3x-2}{\sqrt{5}} \right) + C = \frac{1}{3\sqrt{5}} \tan^{-1} \left(\frac{3x-2}{\sqrt{5}} \right) + C \quad \dots \text{equation 3}$$

From equation 1:

$$I = I_1 - I_2$$

Using equation 2 and equation 3:

$$I = \frac{1}{6} \log|3x^2 - 4x + 3| - \frac{1}{3\sqrt{5}} \tan^{-1} \left(\frac{3x-2}{\sqrt{5}} \right) + C$$

$$I = \frac{1}{6} \log|3x^2 - 4x + 3| - \frac{1}{3\sqrt{5}} \tan^{-1} \left(\frac{3x-2}{\sqrt{5}} \right) + C$$

Exercise 19.20 Page No: 19.106

Evaluate the following integrals:

$$1. \int \frac{x^2 + x + 1}{x^2 - x} dx$$

Solution:

Given $I = \int \frac{x^2+x+1}{x^2-x} dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x) dx + \int \frac{R(x)}{ax^2+bx+c} dx$

$$\Rightarrow \int \frac{x^2 + x + 1}{(x-1)x} dx$$

$$\Rightarrow \int \left(\frac{2x+1}{(x-1)x} + 1 \right) dx$$

$$\Rightarrow \int \frac{2x+1}{(x-1)x} dx + \int 1 dx$$

Consider $\int \frac{2x+1}{(x-1)x} dx$

By partial fraction decomposition,

$$\Rightarrow \frac{2x+1}{(x-1)x} = \frac{A}{x-1} + \frac{B}{x}$$

$$\Rightarrow 2x+1 = Ax + B(x-1)$$

$$\Rightarrow 2x+1 = Ax + Bx - B$$

$$\Rightarrow 2x+1 = (A+B)x - B$$

$$\therefore B = -1 \text{ and } A+B = 2$$

$$\therefore A = 2 + 1 = 3$$

Thus, $\Rightarrow \frac{2x+1}{(x-1)x} = \frac{3}{x-1} - \frac{1}{x}$

$$\Rightarrow \int \left(\frac{3}{x-1} - \frac{1}{x} \right) dx$$

$$\Rightarrow 3 \int \frac{1}{x-1} dx - \int \frac{1}{x} dx$$

Consider $\int \frac{1}{x-1} dx$

Substitute $u = x - 1 \rightarrow dx = du$.

$$\Rightarrow \int \frac{1}{x-1} dx = \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\therefore \int \frac{1}{u} du = \log|u| = \log|x-1|$$

Then,

$$\Rightarrow 3 \int \frac{1}{x-1} dx - \int \frac{1}{x} dx = 3(\log|x-1|) - \int \frac{1}{x} dx$$

$$= 3(\log|x-1|) - \log|x|$$

$$\therefore \int \frac{2x+1}{(x-1)x} dx = 3(\log|x-1|) - \log|x|$$

Then,

$$\Rightarrow \int \frac{2x+1}{(x-1)x} dx + \int 1 dx = 3(\log|x-1|) - \log|x| + \int 1 dx$$

We know that $\int 1 dx = x + c$

$$\Rightarrow \int \frac{2x+1}{(x-1)x} dx + \int 1 dx = 3(\log|x-1|) - \log|x| + x + c$$

$$\therefore I = \int \frac{x^2+x+1}{x^2-x} dx = -\log|x| + x + 3(\log|x-1|) + c$$

$$2. \int \frac{x^2 + x - 1}{x^2 + x - 6} dx$$

Solution:

Consider $I = \int \frac{x^2+x-1}{x^2+x-6} dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x) dx + \int \frac{R(x)}{ax^2+bx+c} dx$

Let $x^2 + x - 1 = x^2 + x - 6 + 5$

$$\Rightarrow \int \frac{x^2 + x - 1}{x^2 + x - 6} dx = \int \left(\frac{x^2 + x - 6}{x^2 + x - 6} + \frac{5}{x^2 + x - 6} \right) dx$$

$$= \int \left(\frac{5}{x^2 + x - 6} + 1 \right) dx$$

$$= 5 \int \left(\frac{1}{x^2 + x - 6} \right) dx + \int 1 dx$$

Consider $\int \frac{1}{x^2+x-6} dx$

Factorizing the denominator,

$$\Rightarrow \int \frac{1}{x^2 + x - 6} dx = \int \frac{1}{(x-2)(x+3)} dx$$

By partial fraction decomposition,

$$\Rightarrow \frac{1}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$$

$$\Rightarrow 1 = A(x+3) + B(x-2)$$

$$\Rightarrow 1 = Ax + 3A + Bx - 2B$$

$$\Rightarrow 1 = (A+B)x + (3A-2B)$$

$$\Rightarrow \text{Then } A+B=0 \dots (1)$$

$$\text{And } 3A-2B=1 \dots (2)$$

Solving (1) and (2),

$$2 \times (1) \rightarrow 2A + 2B = 0$$

$$1 \times (2) \rightarrow 3A - 2B = 1$$

$$5A = 1$$

$$\therefore A = 1/5$$

Substituting A value in (1),

$$\Rightarrow A + B = 0$$

$$\Rightarrow 1/5 + B = 0$$

$$\therefore B = -1/5$$

$$\text{Thus, } \frac{1}{(x-2)(x+3)} = \frac{1}{5(x-2)} - \frac{1}{5(x+3)}$$

$$= \frac{1}{5} \int \frac{1}{x-2} dx - \frac{1}{5} \int \frac{1}{x+3} dx$$

$$\text{Let } x-2 = u \rightarrow dx = du$$

$$\text{And } x+3 = v \rightarrow dx = dv.$$

$$\Rightarrow \frac{1}{5} \int \frac{1}{u} du - \frac{1}{5} \int \frac{1}{v} dv$$

$$\text{We know that } \int \frac{1}{x} dx = \log|x| + c$$

$$\Rightarrow \frac{1}{5} \log|u| - \frac{1}{5} \log|v|$$

$$\Rightarrow \frac{1}{5} \log|x-2| - \frac{1}{5} \log|x+3|$$

$$\Rightarrow \frac{1}{5} (\log|x-2| - \log|x+3|)$$

Then,

$$\Rightarrow 5 \int \left(\frac{1}{x^2+x-6} \right) dx + \int 1 dx = 5 \left(\frac{1}{5} (\log|x-2| - \log|x+3|) \right) + \int 1 dx$$

$$\text{We know that } \int 1 dx = x + c$$

$$\Rightarrow (\log|x-2| - \log|x+3|) + x + c$$

$$\therefore I = \int \frac{x^2+x-1}{x^2+x-6} dx = -\log|x+3| + x + \log|x-2| + c$$

$$\text{Or } I = \log|(x-2)/(x+3)| + x + c$$

$$3. \int \frac{(1-x^2)}{x(1-2x)} dx$$

Solution:

$$\text{Given } I = \int \frac{1-x^2}{(1-2x)x} dx$$

$$\text{Rewriting, we get } \int \frac{x^2-1}{x(2x-1)} dx$$

$$\text{Expressing the integral } \int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x) dx + \int \frac{R(x)}{ax^2+bx+c} dx$$

$$\Rightarrow \int \frac{x^2-1}{x(2x-1)} dx = \int \left(\frac{x-2}{2x(2x-1)} + \frac{1}{2} \right) dx$$

$$= \frac{1}{2} \int \frac{x-2}{x(2x-1)} dx + \frac{1}{2} \int 1 dx$$

$$\text{Consider } \int \frac{x-2}{x(2x-1)} dx$$

By partial fraction decomposition,

$$\Rightarrow \frac{x-2}{x(2x-1)} = \frac{A}{x} + \frac{B}{2x-1}$$

$$\Rightarrow x-2 = A(2x-1) + Bx$$

$$\Rightarrow x-2 = 2Ax - A + Bx$$

$$\Rightarrow x-2 = (2A+B)x - A$$

$$\therefore A = 2 \text{ and } 2A + B = 1$$

$$\therefore B = 1 - 4 = -3$$

$$\text{Thus, } \Rightarrow \frac{x-2}{x(2x-1)} = \frac{2}{x} - \frac{3}{2x-1}$$

$$\Rightarrow \int \left(\frac{2}{x} - \frac{3}{2x-1} \right) dx$$

$$\Rightarrow 2 \int \frac{1}{x} dx - 3 \int \frac{1}{2x-1} dx$$

Consider $\int \frac{1}{x} dx$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{x} dx = \log|x|$$

And consider $\int \frac{1}{2x-1} dx$

Let $u = 2x - 1 \rightarrow dx = 1/2 du$

$$\Rightarrow \int \frac{1}{2x-1} dx = \frac{1}{2} \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \frac{1}{2} \int \frac{1}{u} du = \frac{\log|u|}{2} = \frac{\log|2x-1|}{2}$$

Then,

$$\Rightarrow \int \frac{x-2}{x(2x-1)} dx = 2 \int \frac{1}{x} dx - 3 \int \frac{1}{2x-1} dx$$

$$= 2(\log|x|) - 3 \left(\frac{\log|2x-1|}{2} \right)$$

Then,

$$\Rightarrow \int \frac{x^2-1}{x(2x-1)} dx = \frac{1}{2} \int \frac{x-2}{x(2x-1)} dx + \frac{1}{2} \int 1 dx$$

$$= \frac{1}{2} \left(2(\log|x|) - 3 \left(\frac{\log|2x-1|}{2} \right) \right) + \frac{1}{2} \int 1 \, dx$$

We know that $\int 1 \, dx = x + c$

$$\Rightarrow \log|x| - \frac{3 \log|2x-1|}{4} + \frac{x}{2} + c$$

$$\therefore I = \int \frac{1-x^2}{(1-2x)x} dx = -\frac{3 \log|2x-1|}{4} + \log|x| + \frac{x}{2} + c$$

$$4. \int \frac{x^2 + 1}{x^2 - 5x + 6} dx$$

Solution:

Consider $\int \frac{2x-5}{(x^2-5x+6)} dx$

Let $u = x^2 - 5x + 6 \rightarrow dx = \frac{1}{2x-5} du$

$$\begin{aligned}\Rightarrow \int \frac{2x-5}{(x^2-5x+6)} dx &= \int \frac{2x-5}{u} \frac{1}{2x-5} du \\ &= \int \frac{1}{u} du\end{aligned}$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x^2 - 5x + 6|$$

Now consider $\int \frac{1}{x^2-5x+6} dx$

$$\Rightarrow \int \frac{1}{x^2-5x+6} dx = \int \frac{1}{(x-3)(x-2)} dx$$

By partial fraction decomposition,

$$\Rightarrow \frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$$

$$\Rightarrow 1 = A(x-2) + B(x-3)$$

$$\Rightarrow 1 = Ax - 2A + Bx - 3B$$

$$\Rightarrow 1 = (A+B)x - (2A+3B)$$

$$\Rightarrow A+B=0 \text{ and } 2A+3B=-1$$

Solving the two equations,

$$\Rightarrow 2A+2B=0$$

$$2A+3B=-1$$

$$-B=1$$

$$2A + 3B = -1$$

$$-B = 1$$

$$\therefore B = -1 \text{ and } A = 1$$

$$\Rightarrow \int \frac{1}{(x-3)(x-2)} dx = \int \left(\frac{1}{x-3} - \frac{1}{x-2} \right) dx$$

$$= \int \frac{1}{x-3} dx - \int \frac{1}{x-2} dx$$

$$\text{Consider } \int \frac{1}{x-3} dx$$

$$\text{Let } u = x - 3 \rightarrow dx = du$$

$$\Rightarrow \int \frac{1}{x-3} dx = \int \frac{1}{u} du$$

$$\text{We know that } \int \frac{1}{x} dx = \log|x| + c$$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x-3|$$

$$\text{Similarly } \int \frac{1}{x-2} dx$$

$$\text{Let } u = x - 2 \rightarrow dx = du$$

$$\Rightarrow \int \frac{1}{x-2} dx = \int \frac{1}{u} du$$

$$\text{We know that } \int \frac{1}{x} dx = \log|x| + c$$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x-2|$$

Then,

$$\Rightarrow \int \frac{1}{x^2 - 5x + 6} dx = \int \frac{1}{(x-3)(x-2)} dx = \int \frac{1}{x-3} dx - \int \frac{1}{x-2} dx$$

$$= \log|x-3| - \log|x-2|$$

$$\begin{aligned}
 \Rightarrow \int \frac{x-1}{x^2-5x+6} dx &= \frac{1}{2} \int \frac{2x-5}{(x^2-5x+6)} dx + \frac{3}{2} \int \frac{1}{x^2-5x+6} dx \\
 &= \frac{1}{2} (\log|x^2-5x+6|) + \frac{3}{2} (\log|x-3| - \log|x-2|) \\
 &= \frac{\log|x^2-5x+6|}{2} + \frac{3\log|x-3|}{2} - \frac{3\log|x-2|}{2}
 \end{aligned}$$

Then,

$$\Rightarrow \int \frac{x^2+1}{x^2-5x+6} dx = 5 \int \frac{x-1}{x^2-5x+6} dx + \int 1 dx$$

We know that $\int 1 dx = x + c$

$$\begin{aligned}
 \Rightarrow 5 \int \frac{x-1}{x^2-5x+6} dx + \int 1 dx \\
 &= \frac{5 \log|x^2-5x+6|}{2} + \frac{15 \log|x-3|}{2} - \frac{15 \log|x-2|}{2} + x + c \\
 &= \frac{5 \log|x-2| \log|x-3|}{2} + \frac{15 \log|x-3|}{2} - \frac{15 \log|x-2|}{2} + x + c \\
 &= x - 5 \log|x-2| + 10 \log|x-3| + c
 \end{aligned}$$

$$\therefore I = \int \frac{x^2+1}{x^2-5x+6} dx = x - 5 \log|x-2| + 10 \log|x-3| + c$$

$$5. \int \frac{x^2}{x^2+7x+10} dx$$

Solution:

Given $I = \int \frac{x^2}{x^2+7x+10} dx$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x) dx + \int \frac{R(x)}{ax^2+bx+c} dx$

$$\Rightarrow \int \frac{x^2}{x^2+7x+10} dx = \int \left(\frac{-7x-10}{x^2+7x+10} + 1 \right) dx$$

$$= - \int \frac{7x+10}{x^2+7x+10} dx + \int 1 dx$$

Consider $\int \frac{7x+10}{x^2+7x+10} dx$

Let $7x+10 = \frac{7}{2}(2x+7) - \frac{29}{2}$ and split,

$$\Rightarrow \int \frac{7x+10}{x^2+7x+10} dx = \int \left(\frac{7(2x+7)}{2(x^2+7x+10)} - \frac{29}{2(x^2+7x+10)} \right) dx$$

$$= \frac{7}{2} \int \frac{2x+7}{x^2+7x+10} dx - \frac{29}{2} \int \frac{1}{x^2+7x+10} dx$$

Consider $\int \frac{2x+7}{x^2+7x+10} dx$

Let $u = x^2+7x+10 \rightarrow dx = \frac{1}{2x+7} du$

$$\Rightarrow \int \frac{2x+7}{(x^2+7x+10)} dx = \int \frac{2x+7}{u} \frac{1}{2x+7} du$$

$$= \int \frac{1}{u} du$$

$$= - \int \frac{7x + 10}{x^2 + 7x + 10} dx + \int 1 dx$$

Consider $\int \frac{7x+10}{x^2+7x+10} dx$

Let $7x + 10 = \frac{7}{2}(2x + 7) - \frac{29}{2}$ and split,

$$\Rightarrow \int \frac{7x + 10}{x^2 + 7x + 10} dx = \int \left(\frac{7(2x + 7)}{2(x^2 + 7x + 10)} - \frac{29}{2(x^2 + 7x + 10)} \right) dx$$

$$= \frac{7}{2} \int \frac{2x + 7}{x^2 + 7x + 10} dx - \frac{29}{2} \int \frac{1}{x^2 + 7x + 10} dx$$

Consider $\int \frac{2x+7}{x^2+7x+10} dx$

Let $u = x^2 + 7x + 10 \rightarrow dx = \frac{1}{2x+7} du$

$$\Rightarrow \int \frac{2x + 7}{(x^2 + 7x + 10)} dx = \int \frac{2x + 7}{u} \frac{1}{2x + 7} du$$

$$= \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x^2 + 7x + 10|$$

Now consider $\int \frac{1}{x^2+7x+10} dx$

$$\Rightarrow \int \frac{1}{x^2 + 7x + 10} dx = \int \frac{1}{(x + 2)(x + 5)} dx$$

By partial fraction decomposition,

$$\Rightarrow \frac{1}{(x + 2)(x + 5)} = \frac{A}{x + 2} + \frac{B}{x + 5}$$

$$\Rightarrow 1 = A(x + 2) + B(x + 5)$$

$$\Rightarrow 1 = Ax + 2A + Bx + 5B$$

$$\Rightarrow A + B = 0 \text{ and } 2A + 5B = 1$$

Solving the two equations,

$$\Rightarrow 2A + 2B = 0$$

$$2A + 5B = 1$$

$$-3B = -1$$

$$\therefore B = 1/3 \text{ and } A = -1/3$$

$$\Rightarrow \int \frac{1}{(x+2)(x+5)} dx = \int \left(\frac{-1}{3(x+2)} + \frac{1}{3(x+5)} \right) dx$$

$$= -\frac{1}{3} \int \frac{1}{x+2} dx + \frac{1}{3} \int \frac{1}{x+5} dx$$

$$\text{Consider } \int \frac{1}{x+2} dx$$

$$\text{Let } u = x + 2 \rightarrow dx = du$$

$$\Rightarrow \int \frac{1}{x+2} dx = \int \frac{1}{u} du$$

$$\text{We know that } \int \frac{1}{x} dx = \log|x| + c$$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x+2|$$

$$\text{Similarly } \int \frac{1}{x+5} dx$$

$$\text{Let } u = x + 5 \rightarrow dx = du$$

$$\Rightarrow \int \frac{1}{x+5} dx = \int \frac{1}{u} du$$

$$\text{We know that } \int \frac{1}{x} dx = \log|x| + c$$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x+5|$$

Then,

$$\begin{aligned}\Rightarrow \int \frac{1}{x^2 + 7x + 10} dx &= \int \frac{1}{(x+2)(x+5)} dx = -\frac{1}{3} \int \frac{1}{x+2} dx + \frac{1}{3} \int \frac{1}{x+5} dx \\ &= \frac{-\log|x+2|}{3} + \frac{\log|x+5|}{3}\end{aligned}$$

Then,

$$\begin{aligned}\Rightarrow \int \frac{7x+10}{x^2+7x+10} dx &= \frac{7}{2} \int \frac{2x+7}{x^2+7x+10} dx - \frac{29}{2} \int \frac{1}{x^2+7x+10} dx \\ &= \frac{7}{2} (\log|x^2+7x+10|) - \frac{29}{2} \left(\frac{-\log|x+2|}{3} + \frac{\log|x+5|}{3} \right) \\ &= \frac{7 \log|x^2+7x+10|}{2} + \frac{29 \log|x+2|}{6} - \frac{29 \log|x+5|}{6}\end{aligned}$$

Then,

$$\Rightarrow \int \frac{x^2}{x^2+7x+10} dx = - \int \frac{7x+10}{x^2+7x+10} dx + \int 1 dx$$

We know that $\int 1 dx = x + c$

$$\begin{aligned}\Rightarrow - \int \frac{7x+10}{x^2+7x+10} dx + \int 1 dx \\ &= \frac{-7 \log|x^2+7x+10|}{2} - \frac{29 \log|x+2|}{6} + \frac{29 \log|x+5|}{6} + x + c \\ &= \frac{-7 \log|x+2| \log|x+5|}{2} - \frac{29 \log|x+2|}{6} + \frac{29 \log|x+5|}{6} + x + c\end{aligned}$$

Hence,

$$I = x - \frac{7}{2} \log|x^2+7x+10| + \frac{29}{6} \log \left| \frac{x+2}{x+5} \right| + c$$

Exercise 19.21 Page No: 19.110

Evaluate the following integrals:

$$1. \int \frac{x}{\sqrt{x^2 + 6x + 10}} dx$$

Solution:

Given $I = \int \frac{x}{\sqrt{x^2 + 6x + 10}} dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Writing numerator as $px + q = \lambda \left\{ \frac{d}{dx} (ax^2 + bx + c) \right\} + \mu$

$$\Rightarrow px + q = \lambda (2ax + b) + \mu$$

$$\Rightarrow x = \lambda (2x + 6) + \mu$$

$$\therefore \lambda = 1/2 \text{ and } \mu = -3$$

Let $x = 1/2(2x + 6) - 3$ and split,

$$\begin{aligned} \Rightarrow \int \frac{x}{\sqrt{x^2 + 6x + 10}} dx &= \int \left(\frac{2x + 6}{2\sqrt{x^2 + 6x + 10}} - \frac{3}{\sqrt{x^2 + 6x + 10}} \right) dx \\ &= \int \frac{x + 3}{\sqrt{x^2 + 6x + 10}} dx - 3 \int \frac{1}{\sqrt{x^2 + 6x + 10}} dx \end{aligned}$$

Consider $\int \frac{x+3}{\sqrt{x^2+6x+10}} dx$

Let $u = x^2 + 6x + 10 \rightarrow dx = \frac{1}{2x+6} du$

$$\Rightarrow \int \frac{x + 3}{\sqrt{x^2 + 6x + 10}} dx = \int \frac{1}{2\sqrt{u}} du$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{u}} du$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \frac{1}{2} \int \frac{1}{\sqrt{u}} du = \frac{1}{2} (2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2 + 6x + 10}$$

Consider $\int \frac{1}{\sqrt{x^2 + 6x + 10}} dx$

$$\Rightarrow \int \frac{1}{\sqrt{x^2 + 6x + 10}} dx = \int \frac{1}{\sqrt{(x+3)^2 + 1}} dx$$

Let $u = x + 3 \rightarrow dx = du$

$$\Rightarrow \int \frac{1}{\sqrt{(x+3)^2 + 1}} dx = \int \frac{1}{\sqrt{u^2 + 1}} du$$

We know that $\int \frac{1}{\sqrt{x^2 + 1}} dx = \sinh^{-1} x + c$

$$\Rightarrow \int \frac{1}{\sqrt{u^2 + 1}} du = \sinh^{-1}(u)$$

$$= \sinh^{-1}(x + 3)$$

Then,

$$\Rightarrow \int \frac{x}{\sqrt{x^2 + 6x + 10}} dx = \int \frac{x+3}{\sqrt{x^2 + 6x + 10}} dx - 3 \int \frac{1}{\sqrt{x^2 + 6x + 10}} dx$$

$$= \sqrt{x^2 + 6x + 10} - 3 \sinh^{-1}(x + 3) + c$$

$$\therefore I = \int \frac{x}{\sqrt{x^2 + 6x + 10}} dx = \sqrt{x^2 + 6x + 10} - 3 \sinh^{-1}(x + 3) + c$$

$$2. \int \frac{2x + 1}{\sqrt{x^2 + 2x - 1}} dx$$

Solution:

Given $I = \int \frac{2x+1}{\sqrt{x^2+2x-1}} dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Writing numerator as $px + q = \lambda \left\{ \frac{d}{dx} (ax^2 + bx + c) \right\} + \mu$

$\Rightarrow px + q = \lambda (2ax + b) + \mu$

$\Rightarrow 2x + 1 = \lambda (2x + 2) + \mu$

$\therefore \lambda = 1$ and $\mu = -1$

Let $2x + 1 = 2x + 2 - 1$ and split,

$\Rightarrow \int \frac{2x+1}{\sqrt{x^2+2x-1}} dx = \int \left(\frac{2x+2}{\sqrt{x^2+2x-1}} - \frac{1}{\sqrt{x^2+2x-1}} \right) dx$

$= 2 \int \frac{x+1}{\sqrt{x^2+2x-1}} dx - \int \frac{1}{\sqrt{x^2+2x-1}} dx$

Consider $\int \frac{x+1}{\sqrt{x^2+2x-1}} dx$

Let $u = x^2 + 2x - 1 \rightarrow dx = \frac{1}{2x+2} du$

$\Rightarrow \int \frac{x+1}{\sqrt{x^2+2x-1}} dx = \int \frac{1}{2\sqrt{u}} du$

$= \frac{1}{2} \int \frac{1}{\sqrt{u}} du$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \frac{1}{2} \int \frac{1}{\sqrt{u}} du = \frac{1}{2} (2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2 + 2x - 1}$$

Consider $\int \frac{1}{\sqrt{x^2 + 2x - 1}} dx$

$$\Rightarrow \int \frac{1}{\sqrt{x^2 + 2x - 1}} dx = \int \frac{1}{\sqrt{(x+1)^2 - 2}} dx$$

$$\Rightarrow \int \frac{1}{\sqrt{x^2 + 2x - 1}} dx = \int \frac{1}{\sqrt{(x+1)^2 - 2}} dx$$

$$\text{Let } u = \frac{x+1}{\sqrt{2}} \rightarrow dx = \sqrt{2} du$$

$$\Rightarrow \int \frac{1}{\sqrt{(x+1)^2 - 2}} dx = \int \frac{\sqrt{2}}{\sqrt{2u^2 - 2}} du$$

$$= \int \frac{1}{\sqrt{u^2 - 1}} du$$

$$\text{We know that } \int \frac{1}{\sqrt{x^2 - 1}} dx = \cosh^{-1} x + c$$

$$\Rightarrow \int \frac{1}{\sqrt{u^2 - 1}} du = \cosh^{-1}(u)$$

$$= \cosh^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$

$$\text{Then, } \Rightarrow \int \frac{2x+1}{\sqrt{x^2 + 2x - 1}} dx = 2 \int \frac{x+1}{\sqrt{x^2 + 2x - 1}} dx - \int \frac{1}{\sqrt{x^2 + 2x - 1}} dx$$

$$= 2\sqrt{x^2 + 2x - 1} - \cosh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

$$\therefore I = \int \frac{2x+1}{\sqrt{x^2 + 2x - 1}} dx = 2\sqrt{x^2 + 2x - 1} - \cosh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

$$3. \int \frac{x+1}{\sqrt{4+5x-x^2}} dx$$

Solution:

Given $I = \int \frac{x+1}{\sqrt{4+5x-x^2}} dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Writing numerator as $px + q = \lambda \left\{ \frac{d}{dx} (ax^2 + bx + c) \right\} + \mu$

$\Rightarrow px + q = \lambda (2ax + b) + \mu$

$\Rightarrow x + 1 = \lambda (-2x + 5) + \mu$

$\therefore \lambda = -1/2$ and $\mu = 7/2$

Let $x + 1 = -1/2(-2x + 5) + 7/2$

$$\begin{aligned} \Rightarrow \int \frac{x+1}{\sqrt{-x^2+5x+4}} dx &= \int \left(\frac{-2x+5}{2\sqrt{-x^2+5x+4}} + \frac{7}{2\sqrt{-x^2+5x+4}} \right) dx \\ &= \frac{1}{2} \int \frac{-2x+5}{\sqrt{-x^2+5x+4}} dx + \frac{7}{2} \int \frac{1}{\sqrt{-x^2+5x+4}} dx \end{aligned}$$

Consider $\int \frac{-2x+5}{\sqrt{-x^2+5x+4}} dx$

Let $u = -x^2 + 5x + 4 \rightarrow dx = \frac{1}{-2x+5} du$

$\Rightarrow \int \frac{-2x+5}{\sqrt{-x^2+5x+4}} dx = - \int \frac{1}{\sqrt{u}} du$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

Writing numerator as $px + q = \lambda \left\{ \frac{d}{dx} (ax^2 + bx + c) \right\} + \mu$

$$\Rightarrow px + q = \lambda (2ax + b) + \mu$$

$$\Rightarrow x + 1 = \lambda (-2x + 5) + \mu$$

$$\therefore \lambda = -1/2 \text{ and } \mu = 7/2$$

$$\text{Let } x + 1 = -1/2(-2x + 5) + 7/2$$

$$\Rightarrow \int \frac{x + 1}{\sqrt{-x^2 + 5x + 4}} dx = \int \left(\frac{-2x + 5}{2\sqrt{-x^2 + 5x + 4}} + \frac{7}{2\sqrt{-x^2 + 5x + 4}} \right) dx$$

$$= \frac{1}{2} \int \frac{-2x + 5}{\sqrt{-x^2 + 5x + 4}} dx + \frac{7}{2} \int \frac{1}{\sqrt{-x^2 + 5x + 4}} dx$$

$$\text{Consider } \int \frac{-2x + 5}{\sqrt{-x^2 + 5x + 4}} dx$$

$$\text{Let } u = -x^2 + 5x + 4 \rightarrow dx = \frac{1}{-2x + 5} du$$

$$\Rightarrow \int \frac{-2x + 5}{\sqrt{-x^2 + 5x + 4}} dx = - \int \frac{1}{\sqrt{u}} du$$

$$\text{We know that } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow - \int \frac{1}{\sqrt{u}} du = -(2\sqrt{u})$$

$$= -2\sqrt{x^2 + 6x + 10}$$

$$\Rightarrow \int \frac{x+1}{\sqrt{-x^2+5x+4}} dx = \int \left(\frac{-2x+5}{2\sqrt{-x^2+5x+4}} + \frac{7}{2\sqrt{-x^2+5x+4}} \right) dx$$

$$= \frac{-1}{2} \int \frac{-2x+5}{\sqrt{-x^2+5x+4}} dx + \frac{7}{2} \int \frac{1}{\sqrt{-x^2+5x+4}} dx$$

Consider $\int \frac{-2x+5}{\sqrt{-x^2+5x+4}} dx$

Let $u = -x^2 + 5x + 4 \rightarrow du = -2x+5 dx$

$$\Rightarrow \int \frac{-2x+5}{\sqrt{-x^2+5x+4}} dx = - \int \frac{1}{\sqrt{u}} du$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow - \int \frac{1}{\sqrt{u}} du = -(2\sqrt{u})$$

$$= -2\sqrt{-x^2+5x+4}$$

Consider $\int \frac{1}{\sqrt{-x^2+5x+4}} dx$

$$\Rightarrow \int \frac{1}{\sqrt{-x^2+5x+4}} dx = \int \frac{1}{\sqrt{-\left(x-\frac{5}{2}\right)^2 + \frac{41}{4}}} dx$$

Let $u = \frac{2x-5}{\sqrt{41}} \rightarrow dx = \frac{\sqrt{41}}{2} du$

$$\Rightarrow \int \frac{1}{\sqrt{-\left(x-\frac{5}{2}\right)^2 + \frac{41}{4}}} dx = \int \frac{\sqrt{41}}{\sqrt{41-41u^2}} du$$

$$= \int \frac{1}{\sqrt{1-u^2}} du$$

We know that $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + c$

$$\Rightarrow \int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1}\left(\frac{2x-5}{\sqrt{41}}\right)$$

Then,

$$\Rightarrow \int \frac{x+1}{\sqrt{-x^2+5x+4}} dx = \frac{1}{2} \int \frac{-2x+5}{\sqrt{-x^2+5x+4}} dx + \frac{7}{2} \int \frac{1}{\sqrt{-x^2+5x+4}} dx$$

$$= -\sqrt{-x^2+5x+4} + \frac{7}{2} \left(\sin^{-1}\left(\frac{2x-5}{\sqrt{41}}\right) \right) + c$$

$$\therefore I = \int \frac{x+1}{\sqrt{-x^2+5x+4}} dx = -\sqrt{-x^2+5x+4} + \frac{7}{2} \left(\sin^{-1}\left(\frac{2x-5}{\sqrt{41}}\right) \right) + c$$

$$4. \int \frac{6x-5}{\sqrt{3x^2-5x+1}} dx$$

Solution:

Given $I = \int \frac{6x-5}{\sqrt{3x^2-5x+1}} dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Writing numerator as $px + q = \lambda \left\{ \frac{d}{dx} (ax^2 + bx + c) \right\} + \mu$

$$\Rightarrow px + q = \lambda (2ax + b) + \mu$$

$$\Rightarrow 6x - 5 = \lambda (6x - 5) + \mu$$

$$\therefore \lambda = 1 \text{ and } \mu = 0$$

Let $u = 3x^2 - 5x + 1 \rightarrow dx = \frac{1}{6x-5} du$

$$\Rightarrow \int \frac{6x-5}{\sqrt{3x^2-5x+1}} dx = \int \frac{1}{\sqrt{u}} du$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \int \frac{1}{\sqrt{u}} du = (2\sqrt{u}) + c$$

$$= 2\sqrt{3x^2 - 5x + 1} + c$$

$$\therefore I = \int \frac{6x-5}{\sqrt{3x^2-5x+1}} dx = 2\sqrt{3x^2 - 5x + 1} + c$$

$$\text{Let } u = 3x^2 - 5x + 1 \rightarrow dx = \frac{1}{6x-5} du$$

$$\Rightarrow \int \frac{6x-5}{\sqrt{3x^2-5x+1}} dx = \int \frac{1}{\sqrt{u}} du$$

$$\text{We know that } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow \int \frac{1}{\sqrt{u}} du = (2\sqrt{u}) + c$$

$$= 2\sqrt{3x^2 - 5x + 1} + c$$

$$\therefore I = \int \frac{6x-5}{\sqrt{3x^2-5x+1}} dx = 2\sqrt{3x^2-5x+1} + c$$

$$5. \int \frac{3x+1}{\sqrt{5-2x-x^2}} dx$$

Solution:

Given $I = \int \frac{3x+1}{\sqrt{-x^2-2x+5}} dx$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Writing numerator as $px + q = \lambda \left\{ \frac{d}{dx} (ax^2 + bx + c) \right\} + \mu$

$$\Rightarrow px + q = \lambda (2ax + b) + \mu$$

$$\Rightarrow 3x + 1 = \lambda (-2x - 2) + \mu$$

$$\therefore \lambda = -3/2 \text{ and } \mu = -2$$

$$\text{Let } 3x + 1 = -(3/2) (-2x - 2) - 2$$

$$\Rightarrow \int \frac{3x+1}{\sqrt{-x^2-2x+5}} dx = \int \left(\frac{-3(-2x-2)}{2\sqrt{-x^2-2x+5}} - \frac{2}{\sqrt{-x^2-2x+5}} \right) dx$$

$$= 3 \int \frac{x+1}{\sqrt{-x^2-2x+5}} dx - 2 \int \frac{1}{\sqrt{-x^2-2x+5}} dx$$

Consider $\int \frac{x+1}{\sqrt{-x^2-2x+5}} dx$

$$\text{Let } u = -x^2 - 2x + 5 \rightarrow dx = \frac{1}{-2x-2} du$$

$$\Rightarrow \int \frac{x+1}{\sqrt{-x^2-2x+5}} dx = \int -\frac{1}{2\sqrt{u}} du$$

$$= -\frac{1}{2} \int \frac{1}{\sqrt{u}} du$$

Consider $\int \frac{x+1}{\sqrt{-x^2-2x+5}} dx$

Let $u = -x^2 - 2x + 5 \rightarrow dx = \frac{1}{-2x-2} du$

$$\Rightarrow \int \frac{x+1}{\sqrt{-x^2-2x+5}} dx = \int -\frac{1}{2\sqrt{u}} du$$
$$= -\frac{1}{2} \int \frac{1}{\sqrt{u}} du$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow -\frac{1}{2} \int \frac{1}{\sqrt{u}} du = -(\sqrt{u})$$
$$= -\sqrt{-x^2-2x+5}$$

Consider $\int \frac{1}{\sqrt{-x^2-2x+5}} dx$

$$\Rightarrow \int \frac{1}{\sqrt{-x^2-2x+5}} dx = \int \frac{1}{\sqrt{6-(x+1)^2}} dx$$

Let $u = \frac{x+1}{\sqrt{6}} \rightarrow dx = \sqrt{6} du$

$$\Rightarrow \int \frac{1}{\sqrt{6-(x+1)^2}} dx = \int \frac{\sqrt{6}}{\sqrt{6-6u^2}} du$$

$$= \int \frac{1}{\sqrt{1-u^2}} du$$

We know that $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + c$

$$\Rightarrow \int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)$$

Then,

$$\begin{aligned} \Rightarrow \int \frac{3x+1}{\sqrt{-x^2-2x+5}} dx &= 3 \int \frac{x+1}{\sqrt{-x^2-2x+5}} dx - 2 \int \frac{1}{\sqrt{-x^2-2x+5}} dx \\ &= -3\sqrt{-x^2-2x+5} - 2 \left(\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) \right) + c \end{aligned}$$

$$\begin{aligned} \Rightarrow \int \frac{3x+1}{\sqrt{-x^2-2x+5}} dx &= 3 \int \frac{x+1}{\sqrt{-x^2-2x+5}} dx - 2 \int \frac{1}{\sqrt{-x^2-2x+5}} dx \\ &= -3\sqrt{-x^2-2x+5} - 2 \left(\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) \right) + c \end{aligned}$$

$$\therefore I = \int \frac{3x+1}{\sqrt{-x^2-2x+5}} dx = -3\sqrt{-x^2-2x+5} - 2 \sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + c$$

Exercise 19.22 Page No: 19.114

Evaluate the following integrals:

$$1. \int \frac{1}{4 \cos^2 x + 9 \sin^2 x} dx$$

Solution:

Given $I = \int \frac{1}{4 \cos^2 x + 9 \sin^2 x} dx$

Dividing the numerator and denominator of the given integrand by $\cos^2 x$, we get

$$\Rightarrow I = \int \frac{1}{4 \cos^2 x + 9 \sin^2 x} dx = \int \frac{\sec^2 x}{4 + 9 \tan^2 x} dx$$

Putting $\tan x = t$ and $\sec^2 x dx = dt$, we get

$$\Rightarrow I = \int \frac{dt}{4 + 9t^2} = \frac{1}{9} \int \frac{dt}{\frac{4}{9} + t^2}$$

We know that $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$

$$\Rightarrow \frac{1}{9} \int \frac{dt}{\frac{4}{9} + t^2} = \frac{1}{9} \times \frac{1}{\frac{2}{3}} \tan^{-1} \left(\frac{t}{\frac{2}{3}} \right) + c$$

$$= \frac{1}{6} \tan^{-1} \left(\frac{3t}{2} \right) + c$$

$$= \frac{1}{6} \tan^{-1} \left(\frac{3 \tan x}{2} \right) + c$$

$$\therefore I = \int \frac{1}{4 \cos^2 x + 9 \sin^2 x} dx = \frac{1}{6} \tan^{-1} \left(\frac{3 \tan x}{2} \right) + c$$

2. $\int \frac{1}{4 \sin^2 x + 5 \cos^2 x} dx$

Solution:

Given $I = \int \frac{1}{4 \sin^2 x + 5 \cos^2 x} dx$

Dividing the numerator and denominator of the given integrand by $\cos^2 x$, we get

$$\Rightarrow I = \int \frac{1}{4 \sin^2 x + 5 \cos^2 x} dx = \int \frac{\sec^2 x}{4 \tan^2 x + 5} dx$$

Putting $\tan x = t$ and $\sec^2 x dx = dt$, we get

$$\Rightarrow I = \int \frac{dt}{4t^2 + 5} = \frac{1}{4} \int \frac{dt}{t^2 + \left(\frac{5}{4}\right)}$$

We know that $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$

$$\Rightarrow \frac{1}{4} \int \frac{dt}{t^2 + \left(\frac{5}{4}\right)} = \frac{1}{4} \times \frac{1}{\frac{\sqrt{5}}{2}} \tan^{-1} \left(\frac{t}{\frac{\sqrt{5}}{2}} \right) + c$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1} \left(\frac{2t}{\sqrt{5}} \right) + c$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1} \left(\frac{2 \tan x}{\sqrt{5}} \right) + c$$

$$\therefore I = \int \frac{1}{4 \sin^2 x + 5 \cos^2 x} dx = \frac{1}{2\sqrt{5}} \tan^{-1} \left(\frac{2 \tan x}{\sqrt{5}} \right) + c$$

3. $\int \frac{2}{2 + \sin 2x} dx$

Solution:

Given $I = \int \frac{2}{2 + \sin 2x} dx$

We know that $\sin 2x = 2 \sin x \cos x$

$$\Rightarrow \int \frac{2}{2 + \sin 2x} dx = \int \frac{2}{2 + 2 \sin x \cos x} dx$$

$$= \int \frac{1}{1 + \sin x \cos x} dx$$

Dividing the numerator and denominator by $\cos^2 x$,

$$\Rightarrow \int \frac{1}{1 + \sin x \cos x} dx = \int \frac{\sec^2 x}{\sec^2 x + \tan x} dx$$

Replacing $\sec^2 x$ in denominator by $1 + \tan^2 x$,

$$\Rightarrow \int \frac{\sec^2 x}{\sec^2 x + \tan x} dx = \int \frac{\sec^2 x}{1 + \tan^2 x + \tan x} dx$$

Putting $\tan x = t$ so that $\sec^2 x dx = dt$,

$$\Rightarrow \int \frac{\sec^2 x}{\tan^2 x + \tan x + 1} dx = \int \frac{dt}{t^2 + t + 1}$$

$$= \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

We know that $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$

$$\Rightarrow \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \left(\frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2t + 1}{\sqrt{3}} \right) + c$$

$$\therefore I = \int \frac{2}{2 + \sin 2x} dx = \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2t + 1}{\sqrt{3}} \right) + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2 \tan x + 1}{\sqrt{3}} \right) + C$$

$$4. \int \frac{\cos x}{\cos 3x} dx$$

Solution:

Given $I = \int \frac{\cos x}{\cos 3x} dx$

$$\Rightarrow \int \frac{\cos x}{\cos 3x} dx = \int \frac{\cos x}{4 \cos^3 x - 3 \cos x} dx$$

$$= \int \frac{1}{4 \cos^2 x - 3} dx$$

Dividing numerator and denominator by $\cos^2 x$,

$$\Rightarrow \int \frac{1}{4 \cos^2 x - 3} dx = \int \frac{\sec^2 x}{4 - 3 \sec^2 x} dx$$

Replacing $\sec^2 x$ by $1 + \tan^2 x$ in denominator,

$$\Rightarrow \int \frac{\sec^2 x}{4 - 3 \sec^2 x} dx = \int \frac{\sec^2 x}{4 - 3 - 3 \tan^2 x} dx$$

$$= \int \frac{\sec^2 x}{1 - 3 \tan^2 x} dx$$

Putting $\tan x = t$ and $\sec^2 x dx = dt$, we get

$$I = \int \frac{dt}{1 - 3t^2} = \frac{1}{3} \int \frac{1}{\frac{1}{3} - t^2} dt$$

We know that $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$

$$\Rightarrow \frac{1}{3} \int \frac{1}{\frac{1}{3} - t^2} dt = \frac{1}{3} \times \frac{1}{2\sqrt{3}} \log \left| \frac{\frac{1}{\sqrt{3}} + t}{\frac{1}{\sqrt{3}} - t} \right| + c$$

$$\begin{aligned}
 &\Rightarrow \frac{1}{3} \int \frac{1}{\frac{1}{3} - t^2} dt = \frac{1}{3} \times \frac{1}{2\left(\frac{1}{\sqrt{3}}\right)} \log \left| \frac{\frac{1}{\sqrt{3}} + t}{\frac{1}{\sqrt{3}} - t} \right| + c \\
 &= \frac{1}{2\sqrt{3}} \log \left| \frac{1 + \sqrt{3}t}{1 - \sqrt{3}t} \right| + c \\
 &= \frac{1}{2\sqrt{3}} \log \left| \frac{1 + \sqrt{3} \tan x}{1 - \sqrt{3} \tan x} \right| + c \\
 &\therefore I = \int \frac{\cos x}{\cos 3x} dx = \frac{1}{2\sqrt{3}} \log \left| \frac{1 + \sqrt{3} \tan x}{1 - \sqrt{3} \tan x} \right| + c
 \end{aligned}$$

Exercise 19.23 Page No: 19.117

Evaluate the following integrals:

1. $\int \frac{1}{5 + 4 \cos x} dx$

Solution:

Given $I = \int \frac{1}{5+4\cos x} dx$

We know that $\cos x = \frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}$

$$\Rightarrow \int \frac{1}{5+4\cos x} dx = \int \frac{1}{5+4\left(\frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}\right)} dx$$

$$= \int \frac{1+\tan^2 \frac{x}{2}}{5\left(1+\tan^2 \frac{x}{2}\right)+4\left(1-\tan^2 \frac{x}{2}\right)} dx$$

Replacing $1+\tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1+\tan^2 \frac{x}{2}}{5\left(1+\tan^2 \frac{x}{2}\right)+4\left(1-\tan^2 \frac{x}{2}\right)} dx = \int \frac{\sec^2 \frac{x}{2}}{\tan^2 \frac{x}{2}+9} dx$$

Putting $\tan x/2 = t$ and $\sec^2(x/2) dx = 2dt$,

$$\Rightarrow \int \frac{\sec^2 \frac{x}{2}}{\tan^2 \frac{x}{2}+9} dx = \int \frac{2dt}{t^2+9}$$

$$= 2 \int \frac{1}{t^2+9} dt$$

We know that $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$

$$\Rightarrow 2 \int \frac{1}{t^2+9} dt = 2 \left(\frac{1}{3} \right) \tan^{-1} \left(\frac{t}{3} \right) + c$$

$$= \frac{2}{3} \tan^{-1} \left(\frac{\tan x/2}{3} \right) + c$$

2. $\int \frac{1}{5-4\sin x} dx$

Solution:

Given $I = \int \frac{1}{5-4\sin x} dx$

We know that $\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

$$\Rightarrow \int \frac{1}{5-4\sin x} dx = \int \frac{1}{5-4\left(\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}\right)} dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{5\left(1 + \tan^2 \frac{x}{2}\right) - 4\left(2 \tan \frac{x}{2}\right)} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1 + \tan^2 \frac{x}{2}}{5\left(1 + \tan^2 \frac{x}{2}\right) - 4\left(2 \tan \frac{x}{2}\right)} dx = \int \frac{\sec^2 \frac{x}{2}}{5 + 5 \tan^2 \frac{x}{2} - 8 \tan \frac{x}{2}} dx$$

Putting $\tan x/2 = t$ and $\sec^2(x/2) dx = 2dt$,

$$\Rightarrow \int \frac{\sec^2 \frac{x}{2}}{5 + 5 \tan^2 \frac{x}{2} - 8 \tan \frac{x}{2}} dx = \int \frac{2dt}{5 + 5t^2 - 8t}$$

$$= \frac{2}{5} \int \frac{1}{t^2 - \frac{8}{5}t + 1} dt$$

$$= \frac{2}{5} \int \frac{1}{\left(t - \frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} dt$$

We know that $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$

$$\Rightarrow \frac{2}{5} \int \frac{1}{\left(t - \frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} dt = \frac{2}{5} \left(\frac{1}{\frac{3}{5}} \right) \tan^{-1} \left(\frac{t - \frac{4}{5}}{\frac{3}{5}} \right) + c$$

$$= \frac{2}{3} \tan^{-1} \left(\frac{5 \tan x / 2 - 4}{3} \right) + c$$

3. $\int \frac{1}{1 - 2 \sin x} dx$

Solution:

Given $I = \int \frac{1}{1-2\sin x} dx$

We know that $\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

$$\Rightarrow \int \frac{1}{1-2\sin x} dx = \int \frac{1}{1-2\left(\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}\right)} dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{1\left(1 + \tan^2 \frac{x}{2}\right) - 2\left(2 \tan \frac{x}{2}\right)} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1 + \tan^2 \frac{x}{2}}{1\left(1 + \tan^2 \frac{x}{2}\right) - 2\left(2 \tan \frac{x}{2}\right)} dx = \int \frac{\sec^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2} - 4 \tan \frac{x}{2}} dx$$

Putting $\tan x/2 = t$ and $\sec^2(x/2) dx = 2dt$,

$$\Rightarrow \int \frac{\sec^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2} - 4 \tan \frac{x}{2}} dx = \int \frac{2dt}{1 + t^2 - 4t}$$

$$= 2 \int \frac{1}{t^2 - 4t + 1} dt$$

$$= 2 \int \frac{1}{(t-2)^2 - (\sqrt{3})^2} dt$$

$$= 2 \int \frac{1}{(t-2)^2 - (\sqrt{3})^2} dt$$

We know that $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$

$$\Rightarrow 2 \int \frac{1}{(t-2)^2 - (\sqrt{3})^2} dt = 2 \left(\frac{1}{2\sqrt{3}} \right) \log \left| \frac{(t-2-\sqrt{3})}{(t+2+\sqrt{3})} \right| + c$$

$$= \frac{1}{\sqrt{3}} \log \left| \frac{(\tan x - (2 + \sqrt{3}))}{(\tan x + (2 + \sqrt{3}))} \right| + c$$

$$= \frac{1}{\sqrt{3}} \log \left| \frac{\tan \frac{x}{2} - 2 - \sqrt{3}}{\tan \frac{x}{2} - 2 + \sqrt{3}} \right| + c$$

4. $\int \frac{1}{4 \cos x - 1} dx$

Solution:

Given $I = \int \frac{1}{4 \cos x - 1} dx$

We know that $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

$$\Rightarrow \int \frac{1}{-1 + 4 \cos x} dx = \int \frac{1}{-1 + 4 \left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right)} dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{-1 \left(1 + \tan^2 \frac{x}{2} \right) + 4 \left(1 - \tan^2 \frac{x}{2} \right)} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1 + \tan^2 \frac{x}{2}}{-1 \left(1 + \tan^2 \frac{x}{2} \right) + 4 \left(1 - \tan^2 \frac{x}{2} \right)} dx = \int \frac{\sec^2 \frac{x}{2}}{-5 \tan^2 \frac{x}{2} + 3} dx$$

Putting $\tan \frac{x}{2} = t$ and $\frac{1}{2} \sec^2 \left(\frac{x}{2} \right) dx = dt$,

$$\Rightarrow \int \frac{\sec^2 \frac{x}{2}}{-5 \tan^2 \frac{x}{2} + 3} dx = \int \frac{dt}{3 - 5t^2}$$

$$= \frac{1}{5} \int \frac{1}{\frac{3}{5} - t^2} dt$$

We know that $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$

Putting $\tan \frac{x}{2} = t$ and $\frac{1}{2} \sec^2(\frac{x}{2}) dx = dt$,

$$\Rightarrow \int \frac{\sec^2 \frac{x}{2}}{-5 \tan^2 \frac{x}{2} + 3} dx = \int \frac{2 dt}{3 - 5t^2}$$

$$= \frac{2}{5} \int \frac{1}{\frac{3}{5} - t^2} dt$$

We know that $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$

$$\Rightarrow \frac{2}{5} \int \frac{1}{\frac{3}{5} - t^2} dt = \frac{2}{5} \left(\frac{1}{2\sqrt{\frac{3}{5}}} \right) \log \left| \frac{\sqrt{\frac{3}{5}} + t}{\sqrt{\frac{3}{5}} - t} \right| + c$$

$$= \frac{1}{\sqrt{15}} \log \left| \frac{\sqrt{3} + \sqrt{5} \tan \frac{x}{2}}{\sqrt{3} - \sqrt{5} \tan \frac{x}{2}} \right| + c$$

$$\therefore I = \int \frac{1}{4 \cos x - 1} dx = \frac{1}{\sqrt{15}} \log \left| \frac{\sqrt{3} + \sqrt{5} \tan \frac{x}{2}}{\sqrt{3} - \sqrt{5} \tan \frac{x}{2}} \right| + c$$

5.

$$\int \frac{1}{1 - \sin x + \cos x} dx$$

Solution:

Given $I = \int \frac{1}{1 - \sin x + \cos x} dx$

We know that $\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$ and $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

$$\Rightarrow \int \frac{1}{1 - \sin x + \cos x} dx = \int \frac{1}{1 - \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} + \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}} dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2} - 2 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$ and putting $\tan x/2 = t$ and $\sec^2 x/2 dx = 2dt$,

$$\Rightarrow \int \frac{1 + \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2} - 2 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}} dx = \int \frac{\sec^2 \frac{x}{2}}{2 - 2 \tan \frac{x}{2}} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$ and putting $\tan x/2 = t$ and $\sec^2 x/2 dx = 2dt$,

$$\Rightarrow \int \frac{1 + \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2} - 2 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}} dx = \int \frac{\sec^2 \frac{x}{2}}{2 - 2 \tan \frac{x}{2}} dx$$

$$= \int \frac{2dt}{2 - 2t}$$

$$= \int \frac{1}{1 - t} dt$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{1 - t} dt = -\log|1 - t| + c$$

$$= -\log\left|1 - \tan \frac{x}{2}\right| + c$$

$$\therefore I = \int \frac{1}{1 - \sin x + \cos x} dx = -\log\left|1 - \tan \frac{x}{2}\right| + c$$

Exercise 19.24 Page No: 19.122

Evaluate the following integrals:

1. $\int \frac{1}{1 - \cot x} dx$

Solution:

Let, $I = \int \frac{1}{1 - \cot x} dx$

To solve such integrals involving trigonometric terms in numerator and

denominators. If I has the form $\int \frac{a \sin x + b \cos x + c}{d \sin x + e \cos x + f} dx$

Then substitute numerator as

$$a \sin x + b \cos x + c = A \frac{d}{dx} (d \sin x + e \cos x + f) + B(d \sin x + e \cos x + f) + C$$

Where A, B and C are constants

We have, $I = \int \frac{1}{1 - \cot x} dx = \int \frac{1}{1 - \frac{\cos x}{\sin x}} dx = \int \frac{\sin x}{\sin x - \cos x} dx$

As I matches with the form described above, so we will take the steps as described.

$$\therefore \sin x = A \frac{d}{dx} (\sin x - \cos x) + B(\sin x - \cos x) + C$$

$$\Rightarrow \sin x = A(\cos x + \sin x) + B(\sin x - \cos x) + C \quad \left\{ \because \frac{d}{dx} \cos x = -\sin x \right\}$$

$$\Rightarrow \sin x = \sin x (B + A) + \cos x (A - B) + C$$

Comparing both sides we have:

$$C = 0$$

$$A - B = 0 \Rightarrow A = B$$

$$B + A = 1 \Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$$

$$\therefore A = B = \frac{1}{2}$$

Thus I can be expressed as:

$$I = \int \frac{\frac{1}{2}(\cos x + \sin x) + \frac{1}{2}(\sin x - \cos x)}{\sin x - \cos x} dx$$

$$I = \int \frac{\frac{1}{2}(\cos x + \sin x)}{\sin x - \cos x} dx + \int \frac{\frac{1}{2}(\sin x - \cos x)}{\sin x - \cos x} dx$$

$$\therefore \text{Let } I_1 = \frac{1}{2} \int \frac{(\cos x + \sin x)}{\sin x - \cos x} dx \text{ and } I_2 = \frac{1}{2} \int \frac{(\sin x - \cos x)}{\sin x - \cos x} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$I_1 = \frac{1}{2} \int \frac{(\cos x + \sin x)}{\sin x - \cos x} dx$$

$$\text{Let, } u = \sin x - \cos x \Rightarrow du = (\cos x + \sin x) dx$$

So, I_1 reduces to:

$$I_1 = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \log|u| + C_1$$

$$\therefore I_1 = \frac{1}{2} \log|\sin x - \cos x| + C_1 \dots \text{Equation 2}$$

$$\text{As, } I_2 = \frac{1}{2} \int \frac{(\sin x - \cos x)}{\sin x - \cos x} dx = \frac{1}{2} \int dx$$

$$\therefore I_2 = \frac{x}{2} + C_2 \dots \text{Equation 3}$$

From equation 1, 2 and 3 we have:

$$I = \frac{1}{2} \log|\sin x - \cos x| + C_1 + \frac{x}{2} + C_2$$

$$\therefore I = \frac{1}{2} \log|\sin x - \cos x| + \frac{x}{2} + C$$

$$2. \int \frac{1}{1 - \tan x} dx$$

Solution:

$$\text{Let, } I = \int \frac{1}{1-\tan x} dx$$

To solve such integrals involving trigonometric terms in numerator and denominators. If I has the form $\int \frac{a \sin x + b \cos x + c}{d \sin x + e \cos x + f} dx$

Then substitute numerator as

$$a \sin x + b \cos x + c = A \frac{d}{dx} (d \sin x + e \cos x + f) + B(d \sin x + e \cos x + f) + C$$

Where A, B and C are constants

$$\text{We have, } I = \int \frac{1}{1-\tan x} dx = \int \frac{1}{1-\frac{\sin x}{\cos x}} dx = \int \frac{\cos x}{\cos x - \sin x} dx$$

As I matches with the form described above, so we will take the steps as described.

$$\therefore \cos x = A \frac{d}{dx} (\cos x - \sin x) + B(\cos x - \sin x) + C$$

$$\Rightarrow \cos x = A(-\sin x - \cos x) + B(\cos x - \sin x) + C \left\{ \because \frac{d}{dx} \cos x = -\sin x \right\}$$

$$\Rightarrow \cos x = -\sin x (B + A) + \cos x (B - A) + C$$

Comparing both sides we have:

$$C = 0$$

$$B - A = 1 \Rightarrow A = B - 1$$

$$B + A = 0 \Rightarrow 2B - 1 = 0 \Rightarrow B = \frac{1}{2}$$

$$\therefore A = B - 1 = -\frac{1}{2}$$

Thus I can be expressed as:

$$I = \int \frac{\frac{1}{2}(\cos x + \sin x) + \frac{1}{2}(\cos x - \sin x)}{(\cos x - \sin x)} dx$$

$$I = \int \frac{\frac{1}{2}(\cos x + \sin x)}{(\cos x - \sin x)} dx + \int \frac{\frac{1}{2}(\cos x - \sin x)}{(\cos x - \sin x)} dx$$

$$\therefore \text{Let } I_1 = \frac{1}{2} \int \frac{(\cos x + \sin x)}{(\cos x - \sin x)} dx \text{ and } I_2 = \frac{1}{2} \int \frac{(\cos x - \sin x)}{(\cos x - \sin x)} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$\therefore \text{Let } I_1 = \frac{1}{2} \int \frac{(\cos x + \sin x)}{(\cos x - \sin x)} dx \text{ and } I_2 = \frac{1}{2} \int \frac{(\cos x - \sin x)}{(\cos x - \sin x)} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$I_1 = \frac{1}{2} \int \frac{(\cos x + \sin x)}{(\cos x - \sin x)} dx$$

$$\text{Let, } u = \cos x - \sin x \Rightarrow du = -(\cos x + \sin x) dx$$

So, I_1 reduces to:

$$I_1 = -\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} \log|u| + C_1$$

$$\therefore I_1 = -\frac{1}{2} \log|\cos x - \sin x| + C_1 \dots \text{Equation 2}$$

$$\text{As, } I_2 = \frac{1}{2} \int \frac{(\cos x - \sin x)}{(\cos x - \sin x)} dx = \frac{1}{2} \int dx$$

$$\therefore I_2 = \frac{x}{2} + C_2 \dots \text{Equation 3}$$

From equation 1, 2 and 3 we have:

$$I = -\frac{1}{2} \log|\cos x - \sin x| + C_1 + \frac{x}{2} + C_2$$

$$\therefore I = -\frac{1}{2} \log|\cos x - \sin x| + \frac{x}{2} + C$$

$$3. \int \frac{3 + 2 \cos x + 4 \sin x}{2 \sin x + \cos x + 3} dx$$

Solution:

$$\text{Let, } I = \int \frac{3+2\cos x+4\sin x}{2\sin x+\cos x+3} dx$$

To solve such integrals involving trigonometric terms in numerator and denominators. If I has the form $\int \frac{a \sin x + b \cos x + c}{d \sin x + e \cos x + f} dx$

Then substitute numerator as

$$a \sin x + b \cos x + c = A \frac{d}{dx} (d \sin x + e \cos x + f) + B(d \sin x + e \cos x + f) + C$$

$$a \sin x + b \cos x + c = A \frac{d}{dx} (d \sin x + e \cos x + f) + B(d \sin x + e \cos x + c) + C$$

Where A, B and C are constants

$$\text{We have, } I = \int \frac{3+2 \cos x+4 \sin x}{2 \sin x+\cos x+3} dx$$

As I matches with the form described above, so we will take the steps as described.

$$\therefore 3 + 2 \cos x + 4 \sin x = A \frac{d}{dx} (2 \sin x + \cos x + 3) + B(2 \sin x + \cos x + 3) + C$$

$$\Rightarrow 3 + 2 \cos x + 4 \sin x = A(2 \cos x - \sin x) + B(2 \sin x + \cos x + 3) + C$$

$$\left\{ \because \frac{d}{dx} \cos x = -\sin x \right\}$$

$$\Rightarrow 3 + 2 \cos x + 4 \sin x = \sin x (2B - A) + \cos x (B + 2A) + 3B + C$$

Comparing both sides we have:

$$3B + C = 3$$

$$B + 2A = 2$$

$$2B - A = 4$$

On solving for A, B and C we have:

$$A = 0, B = 2 \text{ and } C = -3$$

Thus I can be expressed as:

$$I = \int \frac{2(2 \sin x + \cos x + 3) - 3}{2 \sin x + \cos x + 3} dx$$

$$I = \int \frac{2(2 \sin x + \cos x + 3)}{2 \sin x + \cos x + 3} dx + \int \frac{-3}{2 \sin x + \cos x + 3} dx$$

$$\therefore \text{Let } I_1 = \int \frac{2(2 \sin x + \cos x + 3)}{2 \sin x + \cos x + 3} dx \text{ and } I_2 = \int \frac{-3}{2 \sin x + \cos x + 3} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$I_1 = \int \frac{2(2 \sin x + \cos x + 3)}{2 \sin x + \cos x + 3} dx$$

So, I_1 reduces to:

$$I_1 = 2 \int \frac{(2 \sin x + \cos x + 3)}{2 \sin x + \cos x + 3} dx$$

So, I_1 reduces to:

$$I_1 = 2 \int dx = 2x + C_1 \dots \text{Equation 2}$$

$$\text{As, } I_2 = -3 \int \frac{1}{2 \sin x + \cos x + 3} dx$$

To solve the integrals of the form $\int \frac{1}{a \sin x + b \cos x + c} dx$

To apply substitution method we take following procedure.

We substitute:

$$\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \text{ and } \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

$$\therefore I_2 = -3 \int \frac{1}{2 \sin x + \cos x + 3} dx$$

$$\Rightarrow I_2 = -3 \int \frac{1}{2 \left(\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right) + 3 \left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right) + 3} dx$$

$$\Rightarrow I_2 = -3 \int \frac{1 + \tan^2 \frac{x}{2}}{4 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2} + 3(1 + \tan^2 \frac{x}{2})} dx$$

$$\Rightarrow I_2 = -3 \int \frac{\sec^2 \frac{x}{2}}{2(2 \tan \frac{x}{2} + 2 + 1 \tan^2 \frac{x}{2})} dx$$

$$\text{Let, } t = \tan \frac{x}{2} \Rightarrow dt = \frac{1}{2} \sec^2 \frac{x}{2} dx$$

$$\therefore I_2 = -3 \int \frac{1}{(2t+2+t^2)} dt$$

As, the denominator is polynomial without any square root term. So one of the special integral will be used to solve I_2 .

$$I_2 = -3 \int \frac{1}{(2t+2+t^2)} dt$$

$$\Rightarrow I_2 = -3 \int \frac{1}{(t^2+2(1)t+1)+1} dt$$

$$\Rightarrow I_2 = -3 \int \frac{1}{(t^2+2(1)t+1)+1} dt$$

$$\therefore I_2 = -3 \int \frac{1}{(t+1)^2+1} dt \quad \{\because a^2 + 2ab + b^2 = (a+b)^2\}$$

As, I_2 matches with the special integral form

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$$I_2 = -3 \tan^{-1}(t+1)$$

Putting value of t we have:

$$\therefore I_2 = -3 \tan^{-1} \left(\tan \frac{x}{2} + 1 \right) + C_2 \dots \dots \text{equation 3}$$

From equation 1, 2 and 3:

$$I = 2x + C_1 - 3 \tan^{-1} \left(\tan \frac{x}{2} + 1 \right) + C_2$$

$$\therefore I = 2x - 3 \tan^{-1} \left(\tan \frac{x}{2} + 1 \right) + C$$

$$4. \int \frac{1}{p + q \tan x} dx$$

Solution:

$$\text{Let, } I = \int \frac{1}{p+q \tan x} dx$$

To solve such integrals involving trigonometric terms in numerator and

denominators. If I has the form $\int \frac{a \sin x + b \cos x + c}{d \sin x + e \cos x + f} dx$

Then substitute numerator as

$$a \sin x + b \cos x + c = A \frac{d}{dx} (d \sin x + e \cos x + f) + B(d \sin x + e \cos x + f) + C$$

Where A, B and C are constants

$$\text{We have, } I = \int \frac{1}{p+q \tan x} dx = \int \frac{1}{p+q \frac{\sin x}{\cos x}} dx = \int \frac{\cos x}{p \cos x + q \sin x} dx$$

As I matches with the form described above, so we will take the form described.

$$\therefore \cos x = A \frac{d}{dx} (p \cos x + q \sin x) + B(p \cos x + q \sin x) + C$$

$$\Rightarrow \cos x = A(-p \sin x + q \cos x) + B(p \cos x + q \sin x) + C \quad \left\{ \because \frac{d}{dx} \cos x = -\sin x \right.$$

$$\Rightarrow \cos x = -\sin x (Bq - Ap) + \cos x (Bp + Aq) + C$$

Comparing both sides we have:

$$C = 0$$

$$Bp + Aq = 1$$

$$Bq - Ap = 0$$

On solving above equations, we have:

$$A = \frac{q}{p^2+q^2} \quad B = \frac{p}{p^2+q^2} \quad \text{and} \quad C = 0$$

Thus I can be expressed as:

$$I = \int \frac{\frac{q}{p^2+q^2} (-p \sin x + q \cos x) + \frac{p}{p^2+q^2} (p \cos x + q \sin x)}{(p \cos x + q \sin x)} dx$$

$$I = \int \frac{\frac{q}{p^2+q^2} (-p \sin x + q \cos x)}{(p \cos x + q \sin x)} dx + \int \frac{\frac{p}{p^2+q^2} (p \cos x + q \sin x)}{(p \cos x + q \sin x)} dx$$

$$\therefore \text{Let } I_1 = \frac{q}{p^2+q^2} \int \frac{(-p \sin x + q \cos x)}{(p \cos x + q \sin x)} dx \quad \text{and} \quad I_2 = \frac{p}{p^2+q^2} \int \frac{(p \cos x + q \sin x)}{(p \cos x + q \sin x)} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$I_1 = \frac{q}{p^2+q^2} \int \frac{(-p \sin x + q \cos x)}{(p \cos x + q \sin x)} dx$$

$$\text{Let, } u = p \cos x + q \sin x \Rightarrow du = (-p \sin x + q \cos x) dx$$

So, I_1 reduces to:

$$I_1 = \frac{q}{p^2+q^2} \int \frac{du}{u} = \frac{q}{p^2+q^2} \log|u| + C_1$$

$$\therefore I_1 = \frac{q}{p^2+q^2} \log|(p \cos x + q \sin x)| + C_1 \text{ Equation 2}$$

$$\text{As, } I_2 = \frac{p}{p^2+q^2} \int \frac{(p \cos x + q \sin x)}{(p \cos x + q \sin x)} dx = \frac{p}{p^2+q^2} \int dx$$

$$\therefore I_2 = \frac{px}{p^2+q^2} + C_2 \text{ Equation 3}$$

From equation 1, 2 and 3 we have:

$$I = \frac{q}{p^2+q^2} \log|(p \cos x + q \sin x)| + C_1 + \frac{px}{p^2+q^2} + C_2$$

$$\therefore I = \frac{q}{p^2+q^2} \log|(p \cos x + q \sin x)| + \frac{px}{p^2+q^2} + C$$

$$5. \int \frac{5 \cos x + 6}{2 \cos x + \sin x + 3} dx$$

Solution:

$$\text{Let, } I = \int \frac{5 \cos x + 6}{2 \cos x + \sin x + 3} dx$$

To solve such integrals involving trigonometric terms in numerator and

denominators. If I has the form $\int \frac{a \sin x + b \cos x + c}{d \sin x + e \cos x + f} dx$

Then substitute numerator as

$$a \sin x + b \cos x + c = A \frac{d}{dx} (d \sin x + e \cos x + f) + B(d \sin x + e \cos x + f) + C$$

Where A, B and C are constants

$$\text{We have, } I = \int \frac{5 \cos x + 6}{2 \cos x + \sin x + 3} dx$$

As I matches with the form described above, so we will take the steps as described.

$$\therefore 5 \cos x + 6 = A \frac{d}{dx} (2 \cos x + \sin x + 3) + B(2 \cos x + \sin x + 3) + C$$

$$\Rightarrow 5 \cos x + 6 = A(-2 \sin x + \cos x) + B(2 \cos x + \sin x + 3) + C$$

$$\therefore \frac{d}{dx} \cos x = -\sin x \}$$

$$\Rightarrow 5 \cos x + 6 = A(-2 \sin x + \cos x) + B(2 \cos x + \sin x + 3) + C$$

$$\because \frac{d}{dx} \cos x = -\sin x \}$$

$$\Rightarrow 5 \cos x + 6 = \sin x (B - 2A) + \cos x (2B + A) + 3B + C$$

Comparing both sides we have:

$$3B + C = 6$$

$$2B + A = 5$$

$$B - 2A = 0$$

On solving for A, B and C we have:

$$A = 1, B = 2 \text{ and } C = 0$$

Thus I can be expressed as:

$$I = \int \frac{(-2 \sin x + \cos x) + 2(2 \cos x + \sin x + 3)}{2 \cos x + \sin x + 3} dx$$

$$I = \int \frac{(-2 \sin x + \cos x)}{2 \cos x + \sin x + 3} dx + \int \frac{2(2 \cos x + \sin x + 3)}{2 \cos x + \sin x + 3} dx$$

$$\therefore \text{Let } I_1 = \int \frac{(-2 \sin x + \cos x)}{2 \cos x + \sin x + 3} dx \text{ and } I_2 = \int \frac{2(2 \cos x + \sin x + 3)}{2 \cos x + \sin x + 3} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$I_1 = \int \frac{(-2 \sin x + \cos x)}{2 \cos x + \sin x + 3} dx$$

$$\text{Let, } 2 \cos x + \sin x + 3 = u$$

$$\Rightarrow (-2\sin x + \cos x) dx = du$$

So, I_1 reduces to:

$$I_1 = \int \frac{du}{u} = \log|u| + C_1$$

$$\therefore I_1 = \log|2 \cos x + \sin x + 3| + C_1 \dots \text{Equation 2}$$

$$\text{As, } I_2 = \int \frac{2(2 \cos x + \sin x + 3)}{2 \cos x + \sin x + 3} dx$$

$$\Rightarrow I_2 = 2 \int dx = 2x + C_2 \dots \text{Equation 3}$$

From equation 1, 2 and 3 we have:

$$\Rightarrow I_2 = 2 \int dx = 2x + C_2 \dots \text{Equation 3}$$

From equation 1, 2 and 3 we have:

$$I = \log|2 \cos x + \sin x + 3| + C_1 + 2x + C_2$$

$$\therefore I = \log|2 \cos x + \sin x + 3| + 2x + C$$

Exercise 19.25 Page No: 19.133

Evaluate the following integrals:

$$1. \int x \cos x dx$$

Solution:

$$\text{Let } I = \int x \cos x \, dx$$

$$\text{We know that, } \int u v \, dx = u \int v \, dx - \int \left(\frac{du}{dx} \int v \, dx \right) dx$$

Using integration by parts,

$$I = x \int \cos x \, dx - \int \left(\frac{d}{dx} x \int \cos x \, dx \right) dx$$

$$\text{We have, } \int \sin x = -\cos x, \int \cos x = \sin x$$

$$= x \times \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x + c$$

$$2. \int \log(x+1) \, dx$$

Solution:

$$\text{Let } I = \int \log(x+1) \, dx$$

That is,

$$I = \int 1 \times \log(x+1) \, dx$$

Using integration by parts,

$$I = \log(x+1) \int 1 \, dx - \int \frac{d}{dx} \log(x+1) \int 1 \, dx$$

We know that, $\int 1 \, dx = x$ and $\int \log x = \frac{1}{x}$

$$= \log(x+1) \times x - \int \frac{1}{x+1} \times x$$

$$\frac{x}{x+1} = 1 - \frac{1}{x+1}$$

$$= x \log(x+1) - \int \left(1 - \frac{1}{x+1}\right) dx$$

$$= x \log(x+1) - x + \log(x+1) + c$$

$$= \log(x+1) \times x - \int \frac{1}{x+1} \times x \, dx$$

Now,

$$\frac{x}{x+1} = 1 - \frac{1}{x+1}$$

$$= x \log(x+1) - \int \left(1 - \frac{1}{x+1}\right) dx$$

$$= x \log(x+1) - x + \log(x+1) + c$$

$$3. \int x^3 \log x \, dx$$

Solution:

$$\text{Let } I = \int x^3 \log x \, dx$$

Using integration by parts,

$$I = \log x \int x^3 \, dx - \int \frac{d}{dx} \log x \int x^3 \, dx$$

$$\text{We have, } \int x^n \, dx = \frac{x^{n+1}}{n+1} \text{ and } \int \log x = \frac{1}{x}$$

$$= \log x \times \frac{x^4}{4} - \int \frac{1}{x} \times \frac{x^4}{4} \, dx$$

$$= \log x \times \frac{x^4}{4} - \frac{1}{4} \int x^3 \, dx$$

$$= \frac{x^4}{4} \log x - \frac{1}{4} \times \frac{x^4}{4}$$

$$= \frac{x^4}{4} \log x - \frac{x^4}{16} + c$$

$$4. \int x e^x \, dx$$

Solution:

$$\text{Let } I = \int x e^x dx$$

Using integration by parts,

$$I = x \int e^x dx - \int \left(\frac{d}{dx} x \int e^x dx \right) dx$$

We know that, $\int e^x dx = e^x$ and $\frac{d}{dx} x = 1$

$$= x e^x - \int e^x dx$$

$$= x e^x - e^x + c$$

$$5. \int x e^{2x} dx$$

Solution:

$$\text{Let } I = \int x e^{2x} dx$$

Using integration by parts,

$$I = x \int e^{2x} dx - \int \left(\frac{d}{dx} x \int e^{2x} dx \right) dx$$

We know that, $\int e^{nx} dx = \frac{e^x}{n}$ and $\frac{d}{dx} x = 1$

$$= \frac{x e^{2x}}{2} - \int \frac{e^{2x}}{2} dx$$

$$= \frac{x e^{2x}}{2} - \frac{e^{2x}}{4} + c$$

$$I = \left(\frac{x}{2} - \frac{1}{4} \right) e^{2x} + c$$

Exercise 19.26 Page No: 19.143

Evaluate the following integrals:

1. $\int e^x (\cos x - \sin x) dx$

Solution:

$$\text{Let } I = \int e^x (\cos x - \sin x) dx$$

Using integration by parts,

$$= \int e^x \cos x dx - \int e^x \sin x dx$$

We know that, $\frac{d}{dx} \cos x = -\sin x$

$$= \cos x \int e^x - \int \frac{d}{dx} \cos x \int e^x dx - \int e^x \sin x dx$$

$$= e^x \cos x + \int e^x \sin x dx - \int e^x \sin x dx$$

$$= e^x \cos x + c$$

$$2. \int e^x \left(\frac{1}{x^2} - \frac{2}{x^3} \right) dx$$

Solution:

$$\text{Let } I = \int e^x \left(\frac{1}{x^2} - \frac{2}{x^3} \right) dx$$

$$= \int e^x x^{-2} dx - 2 \int e^x x^{-3} dx$$

Integrating by parts

$$= x^{-2} \int e^x dx - \int \frac{d}{dx} x^{-2} \int e^x dx - 2 \int e^x x^{-3} dx$$

We know that,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$= e^x x^{-2} + 2 \int e^x x^{-3} dx - 2 \int e^x x^{-3} dx$$

$$= \frac{e^x}{x^2} + c$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$= e^x x^{-2} + 2 \int e^x x^{-3} dx - 2 \int e^x x^{-3} dx$$

$$= \frac{e^x}{x^2} + c$$

$$3. \int e^x \left(\frac{1 + \sin x}{1 + \cos x} \right) dx$$

Solution:

$$\text{Let } I = \int e^x \left(\frac{1+\sin x}{1+\cos x} \right) dx$$

We know that, $\sin^2 x + \cos^2 x = 1$ and $\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$

$$= e^x \left(\frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \right)$$

$$= \frac{e^x \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right)^2}{2 \cos^2 \frac{x}{2}}$$

$$= \frac{1}{2} e^x \left(\frac{\sin \frac{x}{2} + \cos \frac{x}{2}}{\cos \frac{x}{2}} \right)^2$$

$$= \frac{1}{2} e^x \left[\tan \frac{x}{2} + 1 \right]^2$$

$$= \frac{1}{2} e^x \left[1 + \tan \frac{x}{2} \right]^2$$

$$= \frac{1}{2} e^x \left[1 + \tan^2 \frac{x}{2} + 2 \tan \frac{x}{2} \right]$$

$$= \frac{1}{2} e^x \left[\sec^2 \frac{x}{2} + 2 \tan \frac{x}{2} \right]$$

$$= e^x \left[\frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right] \dots \dots (1)$$

Let $\tan \frac{x}{2} = f(x)$

$$f'(x) = \frac{1}{2} \sec^2 \frac{x}{2}$$

We know that,

$$\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + c$$

From equation (1), we obtain

$$\int e^x \left(\frac{1 + \sin x}{1 + \cos x} \right) dx = e^x \tan \frac{x}{2} + c$$

4. $\int e^x (\cot x - \operatorname{cosec}^2 x) dx$

Solution:

$$\text{Let } I = \int e^x (\cot x - \operatorname{cosec}^2 x) dx$$

$$= \int e^x \cot x dx - \int e^x \operatorname{cosec}^2 x dx$$

Integrating by parts,

$$= \cot x \int e^x dx - \int \frac{d}{dx} \cot x \int e^x dx - \int e^x \operatorname{cosec}^2 x dx$$

$$= \cot x e^x + \int e^x \operatorname{cosec}^2 x dx - \int e^x \operatorname{cosec}^2 x dx$$

$$= e^x \cot x + c$$

5. $\int e^x \left(\frac{x-1}{2x^2} \right) dx$

Solution:

Given

$$\int e^x \left(\frac{x-1}{2x^2} \right) dx$$

$$\text{Let } I = \int e^x \frac{1}{2x} dx - \int e^x \frac{1}{2x^2} dx$$

Integrating by parts,

$$= \frac{e^x}{2x} - \int e^x \left(\frac{d}{dx} \left(\frac{1}{2x} \right) \right) dx - \int \frac{e^x}{2x^2} dx$$

$$= \frac{e^x}{2x} + \int \frac{e^x}{2x^2} dx - \int \frac{e^x}{2x^2} dx$$

$$= \frac{e^x}{2x} + c$$

Exercise 19.27 Page No: 19.149

Evaluate the following integrals:

1. $\int e^{ax} \cos bx \, dx$

Solution:

Let $I = \int e^{ax} \cos bx \, dx$

Integrating by parts, we get

$$I = e^{ax} \frac{\sin bx}{b} - a \int e^{ax} \frac{\sin bx}{b} \, dx$$

Taking $1/b$ as common and a/b as common we get

$$= \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \int e^{ax} \sin bx \, dx$$

Now again by using integration by parts, we get

$$= \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \left[-e^{ax} \frac{\cos bx}{b} + a \int e^{ax} \frac{\cos bx}{b} \, dx \right]$$

$$= \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2} \int e^{ax} \cos bx \, dx$$

By computing,

$$I = \frac{e^{ax}}{b^2} [b \sin bx + a \cos bx] - \frac{a^2}{b^2} I + c$$

$$= \frac{e^{ax}}{a^2 + b^2} [b \cos bx + a \sin bx] + c$$

2. $\int e^{ax} \sin(bx + c) \, dx$

Solution:

$$\text{Let } I = \int e^{ax} \sin(bx + c) dx$$

$$= -e^{ax} \frac{\cos(bx + c)}{b} + \int ae^{ax} \frac{\cos(bx + c)}{b} dx$$

Now taking common

$$= -\frac{1}{b} e^{ax} \cos(bx + c) + \frac{a}{b} \int e^{ax} \cos(bx + c)$$

On integrating we get

$$I = \frac{e^{ax}}{b^2} \{a \sin(bx + c) - b \cos(bx + c)\} - \frac{a^2}{b^2} I + c_1$$

By computing the above equation can be written as

$$\begin{aligned} I &= \left\{ \frac{a^2 + b^2}{b^2} \right\} - \frac{e^{ax}}{b^2} \{a \sin(bx + c) - b \cos(bx + c)\} + c_1 \\ &= \frac{e^{ax}}{a^2 + b^2} \{a \sin(bx + c) - b \cos(bx + c)\} + c_1 \end{aligned}$$

$$3. \int \cos(\log x) dx$$

Solution:

$$\text{Let } I = \int \cos(\log x) dx$$

$$\text{Let } \log x = t$$

$$\frac{1}{x} dx = dt$$

$$dx = x dt$$

$$= \int e^t \cos t dt$$

$$\text{We know that, } \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} \{a \sin(bx + c) - b \cos(bx + c)\}$$

$$\text{Hence, } a=1, b=1$$

$$\text{So, } I = \frac{e^t}{2} [\cos t + \sin t] + c$$

Hence,

$$\int \cos(\log x) dx = \frac{e^{\log x}}{2} \{\cos(\log x) + \sin(\log x)\} + c$$

$$I = \frac{x}{2} \{\cos(\log x) + \sin(\log x)\} + c$$

$$4. \int e^{2x} \cos(3x + 4) dx$$

Solution:

$$\text{Let } I = \int e^{2x} \cos(3x + 4) dx$$

Integrating by parts

$$\begin{aligned} I &= e^{2x} \frac{\sin(3x + 4)}{3} - \int 2e^{2x} \frac{\sin(3x + 4)}{3} dx \\ &= \frac{1}{3} e^{2x} \sin(3x + 4) - \frac{2}{3} \int e^{2x} \sin(3x + 4) dx \\ &= \frac{1}{3} e^{2x} \sin(3x + 4) - \frac{2}{3} \left\{ -e^{2x} \frac{\cos(3x + 4)}{3} + \int 2e^{2x} \frac{\cos(3x + 4)}{3} dx \right\} \\ I &= \frac{1}{3} e^{2x} \sin(3x + 4) + \frac{2}{9} e^{2x} \cos(3x + 4) - \frac{4}{9} I \end{aligned}$$

Hence,

$$I = \frac{e^{2x}}{13} [2 \cos(3x + 4) + 3 \sin(3x + 4)] + c$$

$$5. \int e^{2x} \sin x \cos x dx$$

Solution:

$$\text{Let } I = \int e^{2x} \sin x \cos x dx$$

$$= \frac{1}{2} \int e^{2x} 2 \sin x \cos x dx$$

$$= \frac{1}{2} \int e^{2x} \sin 2x dx$$

We know that,

$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} \{a \sin bx - b \cos bx\} + c$$

$$= \frac{e^{2x}}{8} \{2 \sin 2x - 2 \cos 2x\} + c$$

$$I = \frac{1}{2} \frac{e^{2x}}{8} \{2 \sin 2x - 2 \cos 2x\} + c$$

$$I = \frac{e^{2x}}{8} \{\sin 2x - \cos 2x\} + c$$

Exercise 19.28 Page No: 19.154

Evaluate the following integrals:

$$1. \int \sqrt{3 + 2x - x^2} dx$$

Solution:

Let, $I = \int \sqrt{3 + 2x - x^2} \, dx$

$$\therefore I = \int \sqrt{3 - (x^2 - 2(1)x)} \, dx = \int \sqrt{3 - (x^2 - 2(1)x + 1) + 1} \, dx$$

Using $a^2 - 2ab + b^2 = (a - b)^2$

We have:

$$I = \int \sqrt{4 - (x - 1)^2} \, dx = \int \sqrt{2^2 - (x - 1)^2} \, dx$$

As I match with the form:

$$\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

By using above form and simplifying we get

$$\therefore I = \frac{x-1}{2} \sqrt{4 - (x-1)^2} + \frac{4}{2} \sin^{-1} \left(\frac{x-1}{2} \right) + C$$

$$\Rightarrow I = \frac{1}{2} (x-1) \sqrt{3 + 2x - x^2} + 2 \sin^{-1} \left(\frac{x-1}{2} \right) + C$$

2. $\int \sqrt{x^2 + x + 1} \, dx$

Solution:

Let, $I = \int \sqrt{(x^2 + x + 1)} \, dx$

$$\therefore I = \int \sqrt{x^2 + 2 \left(\frac{1}{2} \right) x + \left(\frac{1}{2} \right)^2 + 1 - \left(\frac{1}{2} \right)^2} \, dx$$

Using $a^2 + 2ab + b^2 = (a + b)^2$

We have:

We have:

$$I = \int \sqrt{\left(x + \frac{1}{2}\right)^2 + 1 - \frac{1}{4}} \, dx = \int \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \, dx$$

As I match with the form:

$$\int \sqrt{x^2 + a^2} \, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

$$\therefore I = \frac{\left(x + \frac{1}{2}\right)}{2} \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2} \log \left| \left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right| + C$$

$$\Rightarrow I = \frac{1}{4} (2x + 1) \sqrt{x^2 + x + 1} + \frac{3}{8} \log \left| \left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right| + C$$

$$\Rightarrow I = \frac{1}{4} (2x + 1) \sqrt{x^2 + x + 1} + \frac{3}{8} \log \left| \left(x + \frac{1}{2}\right) + \sqrt{x^2 + x + 1} \right| + C$$

3. $\int \sqrt{x - x^2} \, dx$

Solution:

Let, $I = \int \sqrt{x-x^2} dx$

$$\therefore I = \int \sqrt{-\left(x^2 - 2\left(\frac{1}{2}\right)x\right)} dx = \int \sqrt{\frac{1}{4} - \left(x^2 - 2\left(\frac{1}{2}\right)x + \left(\frac{1}{2}\right)^2\right)} dx$$

Using $a^2 - 2ab + b^2 = (a - b)^2$

We have:

$$I = \int \sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2} dx = \int \sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2} dx$$

As I match with the form: $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + C$

$$\therefore I = \frac{\frac{x-\frac{1}{2}}{2}}{\frac{1}{2}} \sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2} + \frac{\frac{1}{4}}{\frac{1}{2}} \sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right) + C$$

$$\Rightarrow I = \frac{1}{4} (2x - 1) \sqrt{x - x^2} + \frac{1}{8} \sin^{-1}(2x - 1) + C$$

$$\Rightarrow I = \frac{1}{4} (2x - 1) \sqrt{x - x^2} + \frac{1}{8} \sin^{-1}(2x - 1) + C$$

4. $\int \sqrt{1 + x - 2x^2} dx$

Solution:

Let, $I = \int \sqrt{1 + x - 2x^2} \, dx$

$$\therefore I = \int \sqrt{1 - 2\left(x^2 - 2\left(\frac{1}{4}\right)x\right)} \, dx = \int \sqrt{1 - 2\left(x^2 - 2\left(\frac{1}{4}\right)x + \left(\frac{1}{4}\right)^2\right) + 2\left(\frac{1}{4}\right)^2} \, dx$$

Using $a^2 - 2ab + b^2 = (a - b)^2$

We have:

$$I = \int \sqrt{\frac{9}{8} - 2\left(x - \frac{1}{4}\right)^2} \, dx = \int \sqrt{2} \sqrt{\left(\frac{3}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2} \, dx$$

As I match with the form: $\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + C$

$$\therefore I = \sqrt{2} \left\{ \frac{x - \frac{1}{4}}{2} \sqrt{\left(\frac{3}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2} + \frac{9}{2} \sin^{-1}\left(\frac{x - \frac{1}{4}}{\frac{3}{4}}\right) \right\} + C$$

$$\Rightarrow I = \frac{1}{8} (4x - 1) \sqrt{2 \left\{ \left(\frac{3}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2 \right\}} + \frac{9\sqrt{2}}{32} \sin^{-1}\left(\frac{4x-1}{3}\right) + C$$

$$\Rightarrow I = \frac{1}{8} (4x - 1) \sqrt{1 + x - 2x^2} + \frac{9\sqrt{2}}{32} \sin^{-1}\left(\frac{4x-1}{3}\right) + C$$

5. $\int \cos x \sqrt{4 - \sin^2 x} \, dx$

Solution:

Let, $I = \int \cos x \sqrt{4 - \sin^2 x} \, dx$

Let, $\sin x = t$

Differentiating both sides:

$$\Rightarrow \cos x \, dx = dt$$

$$\Rightarrow \cos x \, dx = dt$$

Substituting $\sin x$ with t , we have:

$$\therefore I = \int \sqrt{4 - t^2} \, dt = \int \sqrt{2^2 - t^2} \, dt$$

As I match with the form: $\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$

$$\therefore I = \frac{t}{2} \sqrt{4 - (t)^2} + \frac{4}{2} \sin^{-1} \left(\frac{t}{2} \right) + C$$

Putting the value of t i.e. $t = \sin x$

$$\Rightarrow I = \frac{1}{2} \sin x \sqrt{4 - \sin^2 x} + 2 \sin^{-1} \left(\frac{\sin x}{2} \right) + C$$

Exercise 19.29 Page No: 19.158

Evaluate the following integrals:

$$1. \int (x + 1) \sqrt{x^2 - x + 1} \, dx$$

Solution:

Let us assume $x + 1 = \lambda \frac{d}{dx}(x^2 - x + 1) + \mu$

$$\Rightarrow x + 1 = \lambda \left[\frac{d}{dx}(x^2) - \frac{d}{dx}(x) + \frac{d}{dx}(1) \right] + \mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow x + 1 = \lambda (2x^{2-1} - 1 + 0) + \mu$$

$$\Rightarrow x + 1 = \lambda (2x - 1) + \mu$$

$$\Rightarrow x + 1 = 2\lambda x + \mu - \lambda$$

Comparing the coefficient of x on both sides, we get

$$2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$$

Comparing the constant on both sides, we get

$$\mu - \lambda = 1$$

$$\Rightarrow \mu - \frac{1}{2} = 1$$

$$\therefore \mu = \frac{3}{2}$$

Hence, we have $x + 1 = \frac{1}{2}(2x - 1) + \frac{3}{2}$

Substituting this value in I , we can write the integral as

$$I = \int \left[\frac{1}{2}(2x - 1) + \frac{3}{2} \right] \sqrt{x^2 - x + 1} dx$$

$$\Rightarrow I = \int \left[\frac{1}{2} (2x - 1) \sqrt{x^2 - x + 1} + \frac{3}{2} \sqrt{x^2 - x + 1} \right] dx$$

$$\Rightarrow I = \int \frac{1}{2} (2x - 1) \sqrt{x^2 - x + 1} dx + \int \frac{3}{2} \sqrt{x^2 - x + 1} dx$$

$$\Rightarrow I = \frac{1}{2} \int (2x - 1) \sqrt{x^2 - x + 1} dx + \frac{3}{2} \int \sqrt{x^2 - x + 1} dx$$

$$\text{Let } I_1 = \frac{1}{2} \int (2x - 1) \sqrt{x^2 - x + 1} dx$$

Now, put $x^2 - x + 1 = t$

$$\Rightarrow (2x - 1) dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$I_1 = \frac{1}{2} \int \sqrt{t} dt$$

$$\Rightarrow I_1 = \frac{1}{2} \int t^{\frac{1}{2}} dt$$

$$\text{We know that } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I_1 = \frac{1}{2} \left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{3} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{1}{3} (x^2 - x + 1)^{\frac{3}{2}} + c$$

$$\text{Let } I_2 = \frac{3}{2} \int \sqrt{x^2 - x + 1} dx$$

$$\text{We can write } x^2 - x + 1 = x^2 - 2(x) \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$$

We can write $x^2 - x + 1 = x^2 - 2(x)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$

$$\Rightarrow x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + 1$$

$$\Rightarrow x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\Rightarrow x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

Hence, we can write I_2 as

$$I_2 = \frac{3}{2} \int \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx$$

We know that $\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln|x + \sqrt{x^2 + a^2}| + c$

$$\Rightarrow I_2 = \frac{3}{2} \left[\frac{\left(x - \frac{1}{2}\right)}{2} \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2} \ln \left| \left(x - \frac{1}{2}\right) + \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right| \right] + c$$

$$\Rightarrow I_2 = \frac{3}{2} \left[\frac{2x - 1}{4} \sqrt{x^2 - x + 1} + \frac{3}{8} \ln \left| x - \frac{1}{2} + \sqrt{x^2 - x + 1} \right| \right] + c$$

$$\therefore I_2 = \frac{3}{8} (2x - 1) \sqrt{x^2 - x + 1} + \frac{9}{16} \ln \left| x - \frac{1}{2} + \sqrt{x^2 - x + 1} \right| + c$$

Substituting I_1 and I_2 in I , we get

$$I = \frac{1}{3} (x^2 - x + 1)^{\frac{3}{2}} + \frac{3}{8} (2x - 1) \sqrt{x^2 - x + 1} + \frac{9}{16} \ln \left| x - \frac{1}{2} + \sqrt{x^2 - x + 1} \right| + c$$

Thus,

$$\int (x + 1) \sqrt{x^2 - x + 1} dx = \frac{1}{3} (x^2 - x + 1)^{\frac{3}{2}} + \frac{3}{8} (2x - 1) \sqrt{x^2 - x + 1} + \frac{9}{16} \ln \left| x - \frac{1}{2} + \sqrt{x^2 - x + 1} \right| + c$$

$$2. \int (x + 1) \sqrt{2x^2 + 3} dx$$

Solution:

$$\text{Let } I = \int (x + 1) \sqrt{2x^2 + 3} dx$$

$$\text{Let us assume } x + 1 = \lambda \frac{d}{dx} (2x^2 + 3) + \mu$$

$$\Rightarrow x + 1 = \lambda \left[\frac{d}{dx} (2x^2) + \frac{d}{dx} (3) \right] + \mu$$

$$\Rightarrow x + 1 = \lambda \left[2 \frac{d}{dx} (x^2) + \frac{d}{dx} (3) \right] + \mu$$

$$\text{We know } \frac{d}{dx} (x^n) = nx^{n-1} \text{ and derivative of a constant is 0.}$$

$$\Rightarrow x + 1 = \lambda (2 \times 2x^{2-1} + 0) + \mu$$

$$\Rightarrow x + 1 = \lambda (4x) + \mu$$

$$\Rightarrow x + 1 = 4\lambda x + \mu$$

Comparing the coefficient of x on both sides, we get

$$4\lambda = 1 \Rightarrow \lambda = \frac{1}{4}$$

Comparing the constant on both sides, we get

$$4\lambda = 1 \Rightarrow \lambda = \frac{1}{4}$$

Comparing the constant on both sides, we get

$$\mu = 1$$

$$\text{Hence, we have } x + 1 = \frac{1}{4}(4x) + 1$$

Substituting this value in I, we can write the integral as

$$I = \int \left[\frac{1}{4}(4x) + 1 \right] \sqrt{2x^2 + 3} dx$$

$$\Rightarrow I = \int \left[\frac{1}{4}(4x)\sqrt{2x^2 + 3} + \sqrt{2x^2 + 3} \right] dx$$

$$\Rightarrow I = \int \frac{1}{4} (4x) \sqrt{2x^2 + 3} dx + \int \sqrt{2x^2 + 3} dx$$

$$\Rightarrow I = \frac{1}{4} \int (4x) \sqrt{2x^2 + 3} dx + \int \sqrt{2x^2 + 3} dx$$

$$\text{Let } I_1 = \frac{1}{4} \int (4x) \sqrt{2x^2 + 3} dx$$

$$\text{Now, put } 2x^2 + 3 = t$$

$$\Rightarrow (4x) dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$I_1 = \frac{1}{4} \int \sqrt{t} dt$$

$$\Rightarrow I_1 = \frac{1}{4} \int t^{\frac{1}{2}} dt$$

$$\text{We know that } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I_1 = \frac{1}{4} \left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{4} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{4} \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{6} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{1}{6} (2x^2 + 3)^{\frac{3}{2}} + c$$

$$\text{Let } I_2 = \int \sqrt{2x^2 + 3} dx$$

$$\text{We can write } 2x^2 + 3 = 2 \left(x^2 + \frac{3}{2} \right)$$

$$\Rightarrow 2x^2 + 3 = 2 \left[x^2 + \left(\sqrt{\frac{3}{2}} \right)^2 \right]$$

Hence, we can write I_2 as

$$I_2 = \int \sqrt{2 \left[x^2 + \left(\sqrt{\frac{3}{2}} \right)^2 \right]} dx \Rightarrow I_2 = \sqrt{2} \int \sqrt{x^2 + \left(\sqrt{\frac{3}{2}} \right)^2} dx$$

We know that $\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln|x + \sqrt{x^2 + a^2}| + c$

$$\Rightarrow I_2 = \sqrt{2} \left[\frac{x}{2} \sqrt{x^2 + \left(\sqrt{\frac{3}{2}} \right)^2} + \frac{\left(\sqrt{\frac{3}{2}} \right)^2}{2} \ln \left| x + \sqrt{x^2 + \left(\sqrt{\frac{3}{2}} \right)^2} \right| \right] + c$$

$$\Rightarrow I_2 = \sqrt{2} \left[\frac{x}{2} \sqrt{x^2 + \frac{3}{2}} + \frac{3}{4} \ln \left| x + \sqrt{x^2 + \frac{3}{2}} \right| \right] + c$$

$$\Rightarrow I_2 = \sqrt{2} \left[\frac{x}{2\sqrt{2}} \sqrt{2x^2 + 3} + \frac{3}{2 \times 2} \ln \left| x + \sqrt{x^2 + \frac{3}{2}} \right| \right] + c$$

$$\therefore I_2 = \frac{x}{2} \sqrt{2x^2 + 3} + \frac{3}{2\sqrt{2}} \ln \left| x + \sqrt{x^2 + \frac{3}{2}} \right| + c$$

Substituting I_1 and I_2 in I , we get

$$I = \frac{1}{6} (2x^2 + 3)^{\frac{3}{2}} + \frac{x}{2} \sqrt{2x^2 + 3} + \frac{3}{2\sqrt{2}} \ln \left| x + \sqrt{x^2 + \frac{3}{2}} \right| + c$$

Thus,

$$\int (x+1)\sqrt{2x^2+3}dx = \frac{1}{6} (2x^2 + 3)^{\frac{3}{2}} + \frac{x}{2} \sqrt{2x^2 + 3} + \frac{3}{2\sqrt{2}} \ln \left| x + \sqrt{x^2 + \frac{3}{2}} \right| + c$$

$$3. \int (2x - 5) \sqrt{2 + 3x - x^2} dx$$

Solution:

$$\text{Let } I = \int (2x - 5) \sqrt{2 + 3x - x^2} dx$$

$$\text{Let us assume } 2x - 5 = \lambda \frac{d}{dx} (2 + 3x - x^2) + \mu$$

$$\Rightarrow 2x - 5 = \lambda \left[\frac{d}{dx} (2) + \frac{d}{dx} (3x) - \frac{d}{dx} (x^2) \right] + \mu$$

$$\Rightarrow 2x - 5 = \lambda \left[\frac{d}{dx} (2) + 3 \frac{d}{dx} (x) - \frac{d}{dx} (x^2) \right] + \mu$$

$$\text{We know } \frac{d}{dx} (x^n) = nx^{n-1} \text{ and derivative of a constant is 0.}$$

$$\Rightarrow 2x - 5 = \lambda (0 + 3 - 2x^{2-1}) + \mu$$

$$\Rightarrow 2x - 5 = \lambda (3 - 2x) + \mu$$

$$\Rightarrow 2x - 5 = -2\lambda x + 3\lambda + \mu$$

Comparing the coefficient of x on both sides, we get

$$-2\lambda = 2 \Rightarrow \lambda = -1$$

Comparing the constant on both sides, we get

$$3\lambda + \mu = -5$$

$$\Rightarrow 3(-1) + \mu = -5$$

$$\Rightarrow -3 + \mu = -5$$

$$\therefore \mu = -2$$

$$\text{Hence, we have } 2x - 5 = -(3 - 2x) - 2$$

Substituting this value in I , we can write the integral as

$$I = \int [-(3 - 2x) - 2]\sqrt{2 + 3x - x^2} dx$$

$$\Rightarrow I = \int [-(3 - 2x)\sqrt{2 + 3x - x^2} - 2\sqrt{2 + 3x - x^2}] dx$$

$$\Rightarrow I = - \int (3 - 2x)\sqrt{2 + 3x - x^2} dx - \int 2\sqrt{2 + 3x - x^2} dx$$

$$\Rightarrow I = - \int (3 - 2x)\sqrt{2 + 3x - x^2} dx - \int 2\sqrt{2 + 3x - x^2} dx$$

$$\Rightarrow I = - \int (3 - 2x)\sqrt{2 + 3x - x^2} dx - 2 \int \sqrt{2 + 3x - x^2} dx$$

$$\text{Let } I_1 = - \int (3 - 2x)\sqrt{2 + 3x - x^2} dx$$

Now, put $2 + 3x - x^2 = t$

$$\Rightarrow (3 - 2x) dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$I_1 = - \int \sqrt{t} dt$$

$$\Rightarrow I_1 = - \int t^{\frac{1}{2}} dt$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow I_1 = - \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$\Rightarrow I_1 = - \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = - \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = -\frac{2}{3}t^{\frac{3}{2}} + c$$

$$\therefore I_1 = -\frac{2}{3}(2 + 3x - x^2)^{\frac{3}{2}} + c$$

$$\text{Let } I_2 = -2 \int \sqrt{2 + 3x - x^2} dx$$

$$\text{We can write } 2 + 3x - x^2 = -(x^2 - 3x - 2)$$

$$\Rightarrow 2 + 3x - x^2 = -\left[x^2 - 2(x)\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 2\right]$$

$$\Rightarrow 2 + 3x - x^2 = -\left[\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} - 2\right]$$

$$\Rightarrow 2 + 3x - x^2 = -\left[\left(x - \frac{3}{2}\right)^2 - \frac{17}{4}\right]$$

$$\Rightarrow 2 + 3x - x^2 = \frac{17}{4} - \left(x - \frac{3}{2}\right)^2$$

$$\Rightarrow 2 + 3x - x^2 = \left(\frac{\sqrt{17}}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2$$

Hence, we can write I_2 as

$$I_2 = -2 \int \sqrt{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2} dx$$

$$\Rightarrow 2 + 3x - x^2 = - \left[x^2 - 2(x) \left(\frac{3}{2} \right) + \left(\frac{3}{2} \right)^2 - \left(\frac{3}{2} \right)^2 - 2 \right]$$

$$\Rightarrow 2 + 3x - x^2 = - \left[\left(x - \frac{3}{2} \right)^2 - \frac{9}{4} - 2 \right]$$

$$\Rightarrow 2 + 3x - x^2 = - \left[\left(x - \frac{3}{2} \right)^2 - \frac{17}{4} \right]$$

$$\Rightarrow 2 + 3x - x^2 = \frac{17}{4} - \left(x - \frac{3}{2} \right)^2$$

$$\Rightarrow 2 + 3x - x^2 = \left(\frac{\sqrt{17}}{2} \right)^2 - \left(x - \frac{3}{2} \right)^2$$

Hence, we can write I_2 as

$$I_2 = -2 \int \sqrt{\left(\frac{\sqrt{17}}{2} \right)^2 - \left(x - \frac{3}{2} \right)^2} dx$$

$$\text{We have } \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$$

$$\Rightarrow I_2 = -2 \left[\frac{\left(x - \frac{3}{2} \right)}{2} \sqrt{\left(\frac{\sqrt{17}}{2} \right)^2 - \left(x - \frac{3}{2} \right)^2} + \frac{\left(\frac{\sqrt{17}}{2} \right)^2}{2} \sin^{-1} \left(\frac{x - \frac{3}{2}}{\frac{\sqrt{17}}{2}} \right) \right] + c$$

$$\Rightarrow I_2 = -2 \left[\frac{2x - 3}{4} \sqrt{2 + 3x - x^2} + \frac{17}{8} \sin^{-1} \left(\frac{2x - 3}{\sqrt{17}} \right) \right] + c$$

$$\therefore I_2 = -\frac{1}{2}(2x-3)\sqrt{2+3x-x^2} - \frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right) + c$$

Substituting I_1 and I_2 in I , we get

$$I = -\frac{2}{3}(2+3x-x^2)^{\frac{3}{2}} - \frac{1}{2}(2x-3)\sqrt{2+3x-x^2} - \frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right) + c$$

Thus,

$$\int (2x-5)\sqrt{2+3x-x^2}dx = -\frac{2}{3}(2+3x-x^2)^{\frac{3}{2}} - \frac{1}{2}(2x-3)\sqrt{2+3x-x^2} - \frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right) + c$$

$$\int (2x-5)\sqrt{2+3x-x^2}dx = -\frac{2}{3}(2+3x-x^2)^{\frac{3}{2}} - \frac{1}{2}(2x-3)\sqrt{2+3x-x^2} - \frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right) + c$$

$$4. \int (x+2)\sqrt{x^2+x+1} dx$$

Solution:

$$\text{Let } I = \int (x+2)\sqrt{x^2+x+1}dx$$

$$\text{Let us assume } x+2 = \lambda \frac{d}{dx}(x^2+x+1) + \mu$$

$$\Rightarrow x+2 = \lambda \left[\frac{d}{dx}(x^2) + \frac{d}{dx}(x) + \frac{d}{dx}(1) \right] + \mu$$

$$\text{We know } \frac{d}{dx}(x^n) = nx^{n-1} \text{ and derivative of a constant is 0.}$$

$$\Rightarrow x+2 = \lambda (2x^{2-1} + 1 + 0) + \mu$$

$$\Rightarrow x+2 = \lambda (2x+1) + \mu$$

$$\Rightarrow x+2 = 2\lambda x + \lambda + \mu$$

Comparing the coefficient of x on both sides, we get

$$2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$$

Comparing the constant on both sides, we get

$$\lambda + \mu = 2$$

$$\Rightarrow \frac{1}{2} + \mu = 2$$

$$\therefore \mu = \frac{3}{2}$$

$$\text{Hence, we have } x+2 = \frac{1}{2}(2x+1) + \frac{3}{2}$$

Substituting this value in I , we can write the integral as

$$I = \int \left[\frac{1}{2}(2x+1) + \frac{3}{2} \right] \sqrt{x^2+x+1} dx$$

$$\Rightarrow I = \int \left[\frac{1}{2}(2x+1)\sqrt{x^2+x+1} + \frac{3}{2}\sqrt{x^2+x+1} \right] dx$$

$$\Rightarrow I = \int \frac{1}{2}(2x+1)\sqrt{x^2+x+1} dx + \int \frac{3}{2}\sqrt{x^2+x+1} dx$$

$$\Rightarrow I = \frac{1}{2} \int (2x+1)\sqrt{x^2+x+1} dx + \frac{3}{2} \int \sqrt{x^2+x+1} dx$$

$$\text{Let } I_1 = \frac{1}{2} \int (2x+1)\sqrt{x^2+x+1} dx$$

$$\text{Now, put } x^2+x+1 = t$$

$$\Rightarrow (2x+1) dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$I_1 = \frac{1}{2} \int \sqrt{t} dt$$

$$\Rightarrow I_1 = \frac{1}{2} \int t^{\frac{1}{2}} dt$$

$$\text{We know that } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I_1 = \frac{1}{2} \left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{3} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{1}{3} (x^2 + x + 1)^{\frac{3}{2}} + c$$

$$\text{Let } I_2 = \frac{3}{2} \int \sqrt{x^2 + x + 1} dx$$

$$\text{Let } I_2 = \frac{3}{2} \int \sqrt{x^2 + x + 1} dx$$

$$\text{We can write } x^2 + x + 1 = x^2 + 2(x) \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$$

$$\Rightarrow x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + 1$$

$$\Rightarrow x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\Rightarrow x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

Hence, we can write I_2 as

$$I_2 = \frac{3}{2} \int \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx$$

$$\text{We know that } \int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln|x + \sqrt{x^2 + a^2}| + c$$

$$\Rightarrow I_2 = \frac{3}{2} \left[\frac{\left(x + \frac{1}{2}\right)}{2} \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2} \ln \left| \left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right| \right] + c$$

$$\Rightarrow I_2 = \frac{3}{2} \left[\frac{2x+1}{4} \sqrt{x^2 + x + 1} + \frac{3}{8} \ln \left| x + \frac{1}{2} + \sqrt{x^2 + x + 1} \right| \right] + c$$

$$\therefore I_2 = \frac{3}{8} (2x+1) \sqrt{x^2 + x + 1} + \frac{9}{16} \ln \left| x + \frac{1}{2} + \sqrt{x^2 + x + 1} \right| + c$$

Substituting I_1 and I_2 in I , we get

$$I = \frac{1}{3} (x^2 + x + 1)^{\frac{3}{2}} + \frac{3}{8} (2x+1) \sqrt{x^2 + x + 1} + \frac{9}{16} \ln \left| x + \frac{1}{2} + \sqrt{x^2 + x + 1} \right| + c$$

$$I = \frac{1}{3}(x^2 + x + 1)^{\frac{3}{2}} + \frac{3}{8}(2x + 1)\sqrt{x^2 + x + 1} + \frac{9}{16}\ln\left|x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right| + c$$

Thus,

$$\int (x + 2)\sqrt{x^2 + x + 1} dx = \frac{1}{3}(x^2 + x + 1)^{\frac{3}{2}} + \frac{3}{8}(2x + 1)\sqrt{x^2 + x + 1} + \frac{9}{16}\ln\left|x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right| + c$$

Exercise 19.30 Page No: 19.176

Evaluate the following integrals:

$$1. \int \frac{2x + 1}{(x + 1)(x - 2)} dx$$

Solution:

Here the denominator is already factored.

So let

$$\frac{2x + 1}{(x + 1)(x - 2)} = \frac{A}{x + 1} + \frac{B}{x - 2} \dots\dots (i)$$

$$\Rightarrow \frac{2x + 1}{(x + 1)(x - 2)} = \frac{A(x - 2) + B(x + 1)}{(x + 1)(x - 2)}$$

$$\Rightarrow 2x + 1 = A(x - 2) + B(x + 1) \dots\dots (ii)$$

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put $x = 2$ in the above equation, we get

$$\Rightarrow 2(2) + 1 = A(2 - 2) + B(2 + 1)$$

$$\Rightarrow 3B = 5$$

$$\Rightarrow B = \frac{5}{3}$$

Now put $x = -1$ in equation (ii), we get

$$\Rightarrow 2(-1) + 1 = A((-1) - 2) + B((-1) + 1)$$

$$\Rightarrow -3A = -1$$

$$\Rightarrow A = \frac{1}{3}$$

We put the values of A and B values back into our partial fractions in equation

(i) Now replace this as the integrand. We get

$$\int \left[\frac{A}{x+1} + \frac{B}{x-2} \right] dx$$
$$\Rightarrow \int \left[\frac{\frac{1}{3}}{x+1} + \frac{\frac{5}{3}}{x-2} \right] dx$$

Split up the integral,

$$\Rightarrow \frac{1}{3} \int \left[\frac{1}{x+1} \right] dx + \frac{5}{3} \int \left[\frac{1}{x-2} \right] dx$$

Let substitute $u = x + 1 \Rightarrow du = dx$ and $z = x - 2 \Rightarrow dz = dx$, so the above equation becomes,

$$\Rightarrow \frac{1}{3} \int \left[\frac{1}{u} \right] du + \frac{5}{3} \int \left[\frac{1}{z} \right] dz$$

On integrating we get

$$\Rightarrow \frac{1}{3} \log|u| + \frac{5}{3} \log|z| + C$$

Substituting back, we get

$$\Rightarrow \frac{1}{3} \log|x+1| + \frac{5}{3} \log|x-2| + C$$

The absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{2x+1}{(x+1)(x-2)} dx = \frac{1}{3} \log|x+1| + \frac{5}{3} \log|x-2| + C$$

$$2. \int \frac{1}{x(x-2)(x-4)} dx$$

Solution:

In the given equation the denominator is already factored.

So let

$$\frac{1}{x(x-2)(x-4)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-4} \dots \dots (i)$$

$$\Rightarrow \frac{1}{x(x-2)(x-4)} = \frac{A(x-2)(x-4) + Bx(x-4) + Cx(x-2)}{x(x-2)(x-4)}$$

$$\Rightarrow 1 = A(x-2)(x-4) + Bx(x-4) + Cx(x-2) \dots \dots (ii)$$

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put $x = 0$ in the above equation, we get

$$\Rightarrow 1 = A(0-2)(0-4) + B(0)(0-4) + C(0)(0-2)$$

$$\Rightarrow 1 = 8A + 0 + 0$$

$$\Rightarrow A = \frac{1}{8}$$

Now put $x = 2$ in equation (ii), we get

$$\Rightarrow 1 = A(2-2)(2-4) + B(2)(2-4) + C(2)(2-2)$$

$$\Rightarrow 1 = 0 - 4B + 0$$

$$\Rightarrow B = -\frac{1}{4}$$

Now put $x = 4$ in equation (ii), we get

$$\Rightarrow 1 = A(4-2)(4-4) + B(4)(4-4) + C(4)(4-2)$$

$$\Rightarrow 1 = 0 + 0 + 8C$$

$$\Rightarrow C = \frac{1}{8}$$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\Rightarrow C = \frac{1}{8}$$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-4} \right] dx$$

$$\Rightarrow \int \left[\frac{\frac{1}{8}}{x} + \frac{-\frac{1}{4}}{x-2} + \frac{\frac{1}{8}}{x-4} \right] dx$$

Split up the integral,

$$\Rightarrow \frac{1}{8} \int \left[\frac{1}{x} \right] dx - \frac{1}{4} \int \left[\frac{1}{x-2} \right] dx + \frac{1}{8} \int \left[\frac{1}{x-4} \right] dx$$

Let substitute $u = x - 4 \Rightarrow du = dx$ and $z = x - 2 \Rightarrow dz = dx$, so the above equation becomes,

$$\Rightarrow \frac{1}{8} \int \left[\frac{1}{x} \right] dx - \frac{1}{4} \int \left[\frac{1}{z} \right] dz + \frac{1}{8} \int \left[\frac{1}{u} \right] du$$

On integrating we get

$$\Rightarrow \frac{1}{8} \log|x| - \frac{1}{4} \log|z| + \frac{1}{8} \log|u| + C$$

Substituting back, we get

$$\Rightarrow \frac{1}{8} \log|x| - \frac{1}{4} \log|x-2| + \frac{1}{8} \log|x-4| + C$$

We will take $\frac{1}{8}$ common, we get

$$\Rightarrow \frac{1}{8} [\log|x| - 2\log|x-2| + \log|x-4| + C]$$

Applying the logarithm rule we can rewrite the above equation as

$$\Rightarrow \frac{1}{8} \left[\log \left| \frac{x}{(x-2)^2} \right| + \log|x-4| + C \right]$$

$$\Rightarrow \frac{1}{8} \left[\log \left| \frac{x(x-4)}{(x-2)^2} \right| \right] + C$$

$$\Rightarrow \frac{1}{8} \left[\log \left| \frac{x(x-4)}{(x-2)^2} \right| \right] + C$$

The absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\Rightarrow \frac{1}{8} \left[\log \left| \frac{x(x-4)}{(x-2)^2} \right| \right] + C$$

The absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{1}{x(x-2)(x-4)} dx = \frac{1}{8} \left[\log \left| \frac{x(x-4)}{(x-2)^2} \right| \right] + C$$

$$3. \int \frac{x^2 + x - 1}{x^2 + x - 6} dx$$

Solution:

First we have to simplify numerator, we get

$$\begin{aligned} & \frac{x^2 + x - 1}{x^2 + x - 6} \\ &= \frac{x^2 + x - 6 + 5}{x^2 + x - 6} \\ &= \frac{x^2 + x - 6}{x^2 + x - 6} + \frac{5}{x^2 + x - 6} \\ &= 1 + \frac{5}{x^2 + x - 6} \end{aligned}$$

Now we will factorize denominator by splitting the middle term, we get

$$1 + \frac{5}{x^2 + x - 6}$$

The above equation can be written as

$$= 1 + \frac{5}{x^2 + 3x - 2x - 6}$$

By taking factors common

$$\begin{aligned} &= 1 + \frac{5}{x(x + 3) - 2(x + 3)} \\ &= 1 + \frac{5}{(x + 3)(x - 2)} \end{aligned}$$

Now the denominator is factorized, so let separate the fraction through partial

$$= 1 + \frac{5}{x(x+3) - 2(x+3)}$$

$$= 1 + \frac{5}{(x+3)(x-2)}$$

Now the denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{5}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2} \dots\dots (i)$$

$$\Rightarrow \frac{5}{(x+3)(x-2)} = \frac{A(x-2) + B(x+3)}{(x+3)(x-2)}$$

$$\Rightarrow 5 = A(x-2) + B(x+3) \dots\dots (ii)$$

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put $x = 2$ in the above equation, we get

$$\Rightarrow 5 = A(2-2) + B(2+3)$$

$$\Rightarrow 5 = 0 + 5B$$

$$\Rightarrow B = 1$$

Now put $x = -3$ in equation (ii), we get

$$\Rightarrow 5 = A((-3)-2) + B((-3)+3)$$

$$\Rightarrow 5 = -5A$$

$$\Rightarrow A = -1$$

We put the values of A and B values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\begin{aligned} & \int \left[1 + \frac{A}{x+3} + \frac{B}{x-2} \right] dx \\ & \Rightarrow \int \left[1 + \frac{-1}{x+3} + \frac{1}{x-2} \right] dx \end{aligned}$$

Split up the integral,

$$\begin{aligned} & \Rightarrow \int 1 dx - \int \left[\frac{1}{x+3} \right] dx + \int \left[\frac{1}{x-2} \right] dx \\ & \Rightarrow \int 1 dx - \int \left[\frac{1}{x+3} \right] dx + \int \left[\frac{1}{x-2} \right] dx \end{aligned}$$

Let substitute $u = x + 3 \Rightarrow du = dx$ and $z = x - 2 \Rightarrow dz = dx$, so the above equation becomes,

$$\Rightarrow \int 1 dx - \int \left[\frac{1}{u} \right] du + \int \left[\frac{1}{z} \right] dz$$

On integrating we get

$$\Rightarrow x - \log |u| + \log |z| + C$$

Substituting back, we get

$$\Rightarrow x - \log |x+3| + \log |x-2| + C$$

Applying the logarithm rule, we can rewrite the above equation as

$$\Rightarrow x + \log \left| \frac{x-2}{x+3} \right| + C$$

The absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{x^2 + x - 1}{x^2 + x - 6} dx = x + \log \left| \frac{x-2}{x+3} \right| + C$$

$$4. \int \frac{3 + 4x - x^2}{(x + 2)(x - 1)} dx$$

Solution:

First we simplify numerator, we get

$$\begin{aligned} & \frac{3 + 4x - x^2}{(x + 2)(x - 1)} \\ &= \frac{-(x^2 - 4x - 3)}{x^2 + x - 2} \\ &= \frac{-(x^2 + x - 5x - 2 - 1)}{x^2 + x - 2} \\ &= \frac{-(x^2 + x - 2)}{x^2 + x - 2} + \frac{5x + 1}{x^2 + x - 2} \\ &= -1 + \frac{5x + 1}{(x + 2)(x - 1)} \end{aligned}$$

Now the denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\begin{aligned} \frac{5x + 1}{(x + 2)(x - 1)} &= \frac{A}{x + 2} + \frac{B}{x - 1} \dots \dots (i) \\ \Rightarrow \frac{5x + 1}{(x + 2)(x - 1)} &= \frac{A(x - 1) + B(x + 2)}{(x + 2)(x - 1)} \\ \Rightarrow 5x + 1 &= A(x - 1) + B(x + 2) \dots \dots (ii) \end{aligned}$$

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put $x = 1$ in the above equation, we get

$$\begin{aligned}
 &= \frac{-(x^2 + x - 5x - 2 - 1)}{x^2 + x - 2} \\
 &= \frac{-(x^2 + x - 2)}{x^2 + x - 2} + \frac{5x + 1}{x^2 + x - 2} \\
 &= -1 + \frac{5x + 1}{(x + 2)(x - 1)}
 \end{aligned}$$

Now the denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{5x + 1}{(x + 2)(x - 1)} = \frac{A}{x + 2} + \frac{B}{x - 1} \dots\dots (i)$$

$$\Rightarrow \frac{5x + 1}{(x + 2)(x - 1)} = \frac{A(x - 1) + B(x + 2)}{(x + 2)(x - 1)}$$

$$\Rightarrow 5x + 1 = A(x - 1) + B(x + 2) \dots\dots (ii)$$

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put $x = 1$ in the above equation, we get

$$\Rightarrow 5(1) + 1 = A(1 - 1) + B(1 + 2)$$

$$\Rightarrow 6 = 0 + 3B$$

$$\Rightarrow B = 2$$

Now put $x = -2$ in equation (ii), we get

$$\Rightarrow 5(-2) + 1 = A((-2) - 1) + B((-2) + 2)$$

$$\Rightarrow -9 = -3A + 0$$

$$\Rightarrow A = 3$$

We put the values of A and B values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[-1 + \frac{5x + 1}{(x + 2)(x - 1)} \right] dx$$

$$\Rightarrow \int \left[-1 + \frac{A}{x+2} + \frac{B}{x-1} \right] dx$$

$$\Rightarrow \int \left[-1 + \frac{3}{x+2} + \frac{2}{x-1} \right] dx$$

Split up the integral,

$$\Rightarrow - \int 1 dx + 3 \int \left[\frac{1}{x+2} \right] dx + 2 \int \left[\frac{1}{x-1} \right] dx$$

Let substitute $u = x + 2 \Rightarrow du = dx$ and $z = x - 1 \Rightarrow dz = dx$, so the above equation becomes,

$$\Rightarrow - \int 1 dx + 3 \int \left[\frac{1}{u} \right] du + 2 \int \left[\frac{1}{z} \right] dz$$

On integrating we get

$$\Rightarrow -x + 3 \log|u| + 2 \log|z| + C$$

Substituting back, we get

$$\Rightarrow -x + 3 \log|x + 2| + 2 \log|x - 1| + C$$

The absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{3 + 4x - x^2}{(x+2)(x-1)} dx = -x + 3 \log|x + 2| + 2 \log|x - 1| + C$$

$$5. \int \frac{x^2 + 1}{x^2 - 1} dx$$

Solution:

First we simplify numerator, we get

$$\begin{aligned}\frac{x^2 + 1}{x^2 - 1} \\&= \frac{x^2 - 1 + 2}{x^2 - 1} \\&= \frac{x^2 - 1}{x^2 - 1} + \frac{2}{x^2 - 1} \\&= 1 + \frac{2}{(x - 1)(x + 1)}\end{aligned}$$

Now the denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\begin{aligned}\frac{2}{(x + 1)(x - 1)} &= \frac{A}{x + 1} + \frac{B}{x - 1} \dots\dots (i) \\ \Rightarrow \frac{2}{(x + 2)(x - 1)} &= \frac{A(x - 1) + B(x + 1)}{(x + 2)(x - 1)} \\ \Rightarrow 2 &= A(x - 1) + B(x + 1) \dots\dots (ii)\end{aligned}$$

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put $x = 1$ in the above equation, we get

$$\begin{aligned}\Rightarrow 2 &= A(1 - 1) + B(1 + 1) \\ \Rightarrow 2 &= 0 + 2B\end{aligned}$$

$$= \frac{x^2 - 1 + 2}{x^2 - 1}$$

$$= \frac{x^2 - 1}{x^2 - 1} + \frac{2}{x^2 - 1}$$

$$= 1 + \frac{2}{(x-1)(x+1)}$$

Now the denominator is factorized, so let separate the fraction fraction, hence let

$$\frac{2}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \dots \dots (i)$$

$$\Rightarrow \frac{2}{(x+1)(x-1)} = \frac{A(x-1) + B(x+1)}{(x+1)(x-1)}$$

$$\Rightarrow 2 = A(x-1) + B(x+1) \dots \dots (ii)$$

We need to solve for A and B. One way to do this is to pick values that will cancel each variable.

Put $x = 1$ in the above equation, we get

$$\Rightarrow 2 = A(1-1) + B(1+1)$$

$$\Rightarrow 2 = 0 + 2B$$

$$\Rightarrow B = 1$$

Now put $x = -1$ in equation (ii), we get

$$\Rightarrow 2 = A((-1) - 1) + B((-1) + 1)$$

$$\Rightarrow 2 = -2A + 0$$

$$\Rightarrow A = -1$$

We put the values of A and B values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[1 + \frac{2}{(x-1)(x+1)} \right] dx$$

$$\Rightarrow \int \left[1 + \frac{A}{x+1} + \frac{B}{x-1} \right] dx$$

$$\Rightarrow \int \left[1 + \frac{-1}{x+1} + \frac{1}{x-1} \right] dx$$

Split up the integral,

$$\Rightarrow \int 1 dx - \int \left[\frac{1}{x+1} \right] dx + \int \left[\frac{1}{x-1} \right] dx$$

Let substitute $u = x + 1 \Rightarrow du = dx$ and $z = x - 1 \Rightarrow dz = dx$, so the above equation becomes,

$$\Rightarrow \int 1 dx - \int \left[\frac{1}{u} \right] du + \int \left[\frac{1}{z} \right] dz$$

On integrating we get

$$\Rightarrow x - \log|u| + \log|z| + C$$

Substituting back, we get

$$\Rightarrow x - \log|x+1| + \log|x-1| + C$$

Applying the logarithm rule we get

$$\Rightarrow x + \log \left| \frac{x-1}{x+1} \right| + C$$

The absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{x^2 + 1}{x^2 - 1} dx = x + \log \left| \frac{x-1}{x+1} \right| + C$$

Exercise 19.31 Page No: 19.190

Evaluate the following integrals:

1. $\int \frac{x^2 + 1}{x^4 + x^2 + 1} dx$

Solution:

The given equation can be written as,

$$\int \frac{1 + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$$

$$= \int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 3} dx$$

Let $x - \frac{1}{x} = t$

Then, $\left(1 + \frac{1}{x^2}\right) dx = dt$

$$= \int \frac{1}{t^2 + 3} dt$$

Using standard identity we get

$$= \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{t}{\sqrt{3}} \right) + c$$

Substituting t as $x - \frac{1}{x}$, we get

$$= \frac{1}{\sqrt{3}} \arctan \left(\frac{\left(x - \frac{1}{x}\right)}{\sqrt{3}} \right) + c$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{3}x} \right) + c$$

$$2. \int \sqrt{\cot \theta} d\theta$$

Solution:

Let $\cot \theta$ as x^2

$$-\operatorname{cosec}^2 \theta d\theta = 2x dx$$

$$d\theta = -\frac{2x}{1 + \cot^2 \theta} dx$$

$$d\theta = -\frac{2x}{1 + x^4} dx$$

$$\int -\frac{2x^2}{1 + x^4} dx$$

Re-writing the given equation as

$$\int \frac{1 + \frac{1}{x^2} + 1 - \frac{1}{x^2}}{\frac{1}{x^2} + x^2} dx$$

$$-\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 2} dx - \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 - 2} dx$$

$$\text{Let } x - \frac{1}{x} = t \text{ and } x + \frac{1}{x} = z$$

$$\text{So } \left(1 + \frac{1}{x^2}\right) dx = dt \text{ and } \left(1 - \frac{1}{x^2}\right) dx = dz$$

$$-\int \frac{dt}{(t)^2 + 2} - \int \frac{dz}{(z)^2 - 2}$$

$$-\int \frac{1 + \frac{1}{x^2} + 1 - \frac{1}{x^2}}{\frac{1}{x^2} + x^2} dx$$

$$-\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 2} dx - \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 - 2} dx$$

$$\text{Let } x - \frac{1}{x} = t \text{ and } x + \frac{1}{x} = z$$

$$\text{So } \left(1 + \frac{1}{x^2}\right) dx = dt \text{ and } \left(1 - \frac{1}{x^2}\right) dx = dz$$

$$-\int \frac{dt}{(t)^2 + 2} - \int \frac{dz}{(z)^2 - 2}$$

$$\text{Using identity } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) \text{ and } \int \frac{dz}{(z)^2 - 1} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| +$$

$$-\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}} \log \left| \frac{z - \sqrt{2}}{z + \sqrt{2}} \right| + c$$

Now, substituting t as $x - 1/x$ and z as $x + 1/x$ we have

$$= -\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{2}x} \right) - \frac{1}{2\sqrt{2}} \log \left| \frac{x^2 + 1 - \sqrt{2}x}{x^2 + 1 + \sqrt{2}x} \right| + c$$

Lastly, substituting x^2 as $\cot \theta$ we get

$$I = -\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\cot \theta - 1}{\sqrt{2} \cot \theta} \right) - \frac{1}{2\sqrt{2}} \log \left| \frac{\cot \theta + 1 - \sqrt{2} \cot \theta}{\cot \theta + 1 + \sqrt{2} \cot \theta} \right| + c$$

$$3. \int \frac{x^2 + 9}{x^4 + 81} dx$$

Solution:

$$\int \frac{1 + \frac{9}{x^2}}{x^2 + \frac{81}{x^2}} dx$$

$$= \int \frac{1 + \frac{9}{x^2}}{\left(x - \frac{9}{x}\right)^2 + 18} dx \quad (\text{By completing the square})$$

$$\text{Let } x - \frac{9}{x} = t$$

$$\left(1 + \frac{9}{x^2}\right) dx = dt$$

$$= \int \frac{dt}{t^2 + 18}$$

$$\text{Using identity } \int \frac{1}{x^2 + 1} dx = \tan^{-1}(x)$$

$$= \frac{1}{3\sqrt{2}} \tan^{-1}\left(\frac{t}{3\sqrt{2}}\right) + c$$

$$\text{Substituting } t \text{ as } x - \frac{1}{x}$$

$$= \frac{1}{3\sqrt{2}} \tan^{-1}\left(\frac{x - \frac{1}{x}}{3\sqrt{2}}\right) + c$$

4. $\int \frac{1}{x^4 + x^2 + 1} dx$

Solution:

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$$\int \frac{\frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$$

$$= \frac{1}{2} \int \frac{1 + \frac{1}{x^2} + \frac{1}{x^2} - 1}{x^2 + 1 + \frac{1}{x^2}} dx \quad \text{(Manipulating the numerator by multiplying and dividing by 2)}$$

$$= \frac{1}{2} \left[\int \frac{1 + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx + \int \frac{-1 + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx \right]$$

$$= \frac{1}{2} \left[\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 3} dx + \int \frac{-1 + \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 - 1} dx \right]$$

$$\text{Let } x - \frac{1}{x} = t \text{ and } x + \frac{1}{x} = z$$

$$\text{Then, } \left(1 + \frac{1}{x^2}\right) dx = dt \text{ and } \left(1 - \frac{1}{x^2}\right) dx = dz$$

$$= \frac{1}{2} \left[\int \frac{dt}{(t)^2 + 3} - \int \frac{dz}{(z)^2 - 1} \right]$$

$$\text{Using identity } \int \frac{1}{x^2 + 1} dx = \tan^{-1}(x) \text{ and } \int \frac{dz}{(z)^2 - 1} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$$

$$= \frac{1}{2} \left[\frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{t}{\sqrt{3}} \right) - \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| \right]$$

We get,

$$= \frac{1}{2} \left[\frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x - \frac{1}{x}}{\sqrt{3}} \right) - \frac{1}{2} \log \left| \frac{x + \frac{1}{x} - 1}{x + \frac{1}{x} + 1} \right| \right] + c$$

$$I = \frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{3}x} \right) - \frac{1}{4} \log \left| \frac{x^2 + 1 - x}{x^2 + 1 + x} \right| + c$$

5. $\int \frac{x^2 - 3x + 1}{x^4 + x^2 + 1} dx$

Solution:

The given equation can be written as

$$\int \frac{1 - \frac{3}{x} + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$$

$$\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 3} dx - \int \frac{3x}{x^4 + x^2 + 1} dx$$

Substituting t as $x - \frac{1}{x}$ and z as x^2

$$\left(1 + \frac{1}{x^2}\right) dx = dt \quad \text{And } 2x dx = dz$$

$$\int \frac{dt}{(t)^2 + 3} - \frac{3}{2} \int \frac{dz}{z^2 + z + 1}$$

$$\int \frac{dt}{(t)^2 + 3} - \frac{3}{2} \int \frac{dz}{\left(z + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

Using identity $\int \frac{1}{x^2 + 1} dx = \tan^{-1}(x)$

$$\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) - \sqrt{3} \tan^{-1}\left(\frac{2z + 1}{\sqrt{3}}\right) + c$$

Substituting t as $x - \frac{1}{x}$ and z as x^2

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x - \frac{1}{x}}{\sqrt{3}}\right) - \sqrt{3} \tan^{-1}\left(\frac{2x^2 + 1}{\sqrt{3}}\right) + c$$

Hence,

$$I = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{3}x}\right) - \sqrt{3} \tan^{-1}\left(\frac{2x^2 + 1}{\sqrt{3}}\right) + c$$

Exercise 19.32 Page No: 19.196

Evaluate the following integrals:

1. $\int \frac{1}{(x-1)\sqrt{x+2}} dx$

Solution:

Assume $x + 2 = t^2$

$dx = 2t dt$

Now, $\int \frac{2dt}{(t^2 - 3)}$

Using identity $\int \frac{dz}{(z)^2 - 1} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$

$= \frac{1}{\sqrt{3}} \log \left| \frac{t - \sqrt{3}}{t + \sqrt{3}} \right| + c$

$= \frac{1}{\sqrt{3}} \log \left| \frac{\sqrt{(x+2)} - \sqrt{3}}{\sqrt{x+2} + \sqrt{3}} \right| + c$

2. $\int \frac{1}{(x-1)\sqrt{2x+3}} dx$

Solution:

Assume $2x + 3 = t^2$

$dx = t \, dt$

$$\int \frac{dt}{\frac{t^2 - 3}{2} - 1}$$

$$\int \frac{2dt}{(t^2 - 5)}$$

Using identity $\int \frac{dz}{(z)^2 - 1} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$

$$\frac{1}{\sqrt{5}} \log \left| \frac{t - \sqrt{5}}{t + \sqrt{5}} \right| + c$$

$$\frac{1}{\sqrt{5}} \log \left| \frac{\sqrt{(2x+3)} - \sqrt{5}}{\sqrt{2x+3} + \sqrt{5}} \right| + c$$

$$\frac{1}{\sqrt{5}} \log \left| \frac{t - \sqrt{5}}{t + \sqrt{5}} \right| + c$$

$$\frac{1}{\sqrt{5}} \log \left| \frac{\sqrt{(2x+3)} - \sqrt{5}}{\sqrt{2x+3} + \sqrt{5}} \right| + c$$

3. $\int \frac{x+1}{(x-1)\sqrt{x+2}} \, dx$

Solution:

The given equation can be written as

$$\int \frac{(x-1)+2}{(x-1)\sqrt{x+2}} dx$$

Now splitting the integral in two parts

$$\int \frac{(x-1)}{(x-1)\sqrt{x+2}} dx + \int \frac{2}{(x-1)\sqrt{x+2}} dx$$

For the first part using identity $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$2\sqrt{x+2}$$

For the second part

Assume $x+2=t^2$

$$dx = 2t dt$$

$$\int \frac{4dt}{(t^2-3)}$$

Using identity $\int \frac{dz}{(z)^2-1} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$

$$\frac{2}{\sqrt{3}} \log \left| \frac{t-\sqrt{3}}{t+\sqrt{3}} \right| + c$$

$$\frac{2}{\sqrt{3}} \log \left| \frac{\sqrt{(x+2)}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}} \right| + c$$

Hence integral is

$$2\sqrt{x+2} + \frac{2}{\sqrt{3}} \log \left| \frac{\sqrt{(x+2)}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}} \right| + c$$

$$4. \int \frac{x^2}{(x-1)\sqrt{x+2}} dx$$

Solution:

The given equation can be written as

$$\int \frac{(x^2 - 1) + 1}{(x - 1)\sqrt{x + 2}} dx$$

$$\int \frac{(x^2 - 1)}{(x - 1)\sqrt{x + 2}} dx + \int \frac{1}{(x - 1)\sqrt{x + 2}} dx$$

$$\int \frac{(x + 1)}{\sqrt{x + 2}} dx + \int \frac{1}{(x - 1)\sqrt{x + 2}} dx$$

$$\int \frac{(1)}{\sqrt{x + 2}} dx + \int \sqrt{x + 2} dx + \int \frac{1}{(x - 1)\sqrt{x + 2}} dx$$

For the first- and second-part using identity $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$\frac{2}{3}(x + 2)^{\frac{3}{2}} + 2\sqrt{x + 2}$$

For the second part

Assume $x+2=t^2$

$$dx = 2t dt$$

$$\int \frac{4dt}{(t^2 - 3)}$$

Using identity $\int \frac{dz}{(z)^2 - 1} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$

$$\int \frac{(x+1)}{\sqrt{x+2}} dx + \int \frac{1}{(x-1)\sqrt{x+2}} dx = \int \frac{(x+2)-1}{\sqrt{x+2}} dx + \int \frac{1}{(x-1)\sqrt{x+2}} dx$$

$$\int \frac{(1)}{\sqrt{x+2}} dx + \int \sqrt{x+2} dx + \int \frac{1}{(x-1)\sqrt{x+2}} dx$$

For the first- and second-part using identity $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$\frac{2}{3}(x+2)^{\frac{3}{2}} - 2\sqrt{x+2}$$

For the second part

Assume $x+2=t^2$

$$dx = 2t dt$$

So,

$$\int \frac{2dt}{(t^2-3)}$$

Using identity $\int \frac{dz}{(z)^2-1} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$

$$= 2 \cdot \frac{1}{2\sqrt{3}} \log \left| \frac{t - \sqrt{3}}{t + \sqrt{3}} \right| + c$$

$$= \frac{1}{\sqrt{3}} \log \left| \frac{\sqrt{(x+2)} - \sqrt{3}}{\sqrt{x+2} + \sqrt{3}} \right| + c \quad (\text{Using } t^2 = x + 2)$$

Hence integral is

$$I = \frac{2}{3} (x+2)^{\frac{3}{2}} - 2\sqrt{x+2} + \frac{1}{\sqrt{3}} \log \left| \frac{\sqrt{(x+2)} - \sqrt{3}}{\sqrt{x+2} + \sqrt{3}} \right| + c$$

5. $\int \frac{x}{(x-3)\sqrt{x+1}} dx$

Solution:

The given equation can be written as

$$\int \frac{(x-3)+3}{(x-3)\sqrt{x+1}} dx$$

$$\int \frac{(x-3)}{(x-3)\sqrt{x+1}} dx + \int \frac{3}{(x-3)\sqrt{x+1}} dx$$

For the first part using identity $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$2\sqrt{x+1} + c$$

For the second part $\int \frac{3}{(x-3)\sqrt{x+1}} dx,$

Assume $x+1 = t^2$

$$dx = 2t dt$$

$$\int \frac{3 \cdot 2dt}{(t^2-4)}$$

Using identity $\int \frac{dz}{(z)^2-1} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$

$$= 3 \times 2 \times \frac{1}{2(2)} \log \left| \frac{t-2}{t+2} \right| + c$$

$$= \frac{3}{2} \log \left| \frac{\sqrt{(x+2)} - 2}{\sqrt{x+2} + 2} \right| + c$$

Hence integral is

$$= 2\sqrt{x+1} + \frac{3}{2} \log \left| \frac{\sqrt{(x+2)} - 2}{\sqrt{x+2} + 2} \right| + c$$