

RD Sharma Solutions for Class 11 Maths Chapter 10 Sine and Cosine Formulae and their Applications

1. If in a $\triangle ABC$, $\angle A = 45^\circ$, $\angle B = 60^\circ$, and $\angle C = 75^\circ$; find the ratio of its sides.

Solution:

Given: In $\triangle ABC$, $\angle A = 45^\circ$, $\angle B = 60^\circ$, and $\angle C = 75^\circ$

By using the sine rule, we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Now by substituting the values we get,

$$\frac{a}{\sin 45^\circ} = \frac{b}{\sin 60^\circ} = \frac{c}{\sin 75^\circ}$$

$$\frac{a}{\sin 45^\circ} = \frac{b}{\sin 60^\circ} = \frac{c}{\sin(30^\circ + 45^\circ)}$$

$$\frac{a}{\sin 45^\circ} = \frac{b}{\sin 60^\circ} = \frac{c}{\sin 30^\circ \cos 45^\circ + \sin 45^\circ \cos 30^\circ}$$

We know, $\sin(a + b) = \sin a \cos b + \sin b \cos a$

Now by substituting the corresponding values, we get,

$$\frac{a}{\frac{1}{\sqrt{2}}} = \frac{b}{\frac{\sqrt{3}}{2}} = \frac{c}{\frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}}$$

$$\frac{a}{\frac{1}{\sqrt{2}}} = \frac{b}{\frac{\sqrt{3}}{2}} = \frac{c}{\frac{1 + \sqrt{3}}{2\sqrt{2}}}$$

$$a:b:c = \frac{1}{\sqrt{2}} : \frac{\sqrt{3}}{2} : \frac{1 + \sqrt{3}}{2\sqrt{2}}$$

Multiply the above expression by $2\sqrt{2}$, we get

$$a:b:c = 2:\sqrt{6}:(1+\sqrt{3})$$

Hence the ratio of the sides of the given triangle is $a:b:c = 2:\sqrt{6}:(1+\sqrt{3})$

2. If in any $\triangle ABC$, $\angle C = 105^\circ$, $\angle B = 45^\circ$, $a = 2$, then find b .

Solution:

Given: In $\triangle ABC$, $\angle C = 105^\circ$, $\angle B = 45^\circ$, $a = 2$

We know in a triangle,

$$\angle A + \angle B + \angle C = 180^\circ$$

$$\angle A = 180^\circ - \angle B - \angle C$$

Substituting the given values, we get

$$\angle A = 180^\circ - 45^\circ - 105^\circ$$

$$\angle A = 30^\circ$$

By using the sine rule, we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Now by substituting the corresponding values we get,

$$\frac{2}{\sin 30^\circ} = \frac{b}{\sin 45^\circ}$$

Substitute the equivalent values of the sine, we get

$$\frac{2}{\frac{1}{2}} = \frac{b}{\frac{1}{\sqrt{2}}}$$

$$4 = b\sqrt{2}$$

$$b = 4/\sqrt{2}$$

$$= 2\sqrt{2}$$

Hence the value of b is $2\sqrt{2}$ units.

3. In $\triangle ABC$, if $a = 18$, $b = 24$ and $c = 30$ and $\angle C = 90^\circ$, find $\sin A$, $\sin B$ and $\sin C$.

Solution:

Given: In $\triangle ABC$, $a = 18$, $b = 24$ and $c = 30$ and $\angle C = 90^\circ$

By using the sine rule, we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Now by substituting the given values we get,

$$\frac{18}{\sin A} = \frac{30}{\sin 90^\circ}$$

$$\sin A = \frac{18 \times \sin 90^\circ}{30}$$

$$\sin A = \frac{18 \times 1}{30}$$

$$\sin A = \frac{3}{5}$$

Similarly,

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Substitute the given values, we get

$$\frac{24}{\sin B} = \frac{30}{\sin 90^\circ}$$

$$\sin B = \frac{24 \times \sin 90^\circ}{30}$$

$$\sin B = \frac{24 \times 1}{30}$$

$$\sin B = \frac{4}{5}$$

And given, $\angle C = 90^\circ$, so $\sin C = \sin 90^\circ = 1$.

Hence the values of $\sin A = 3/5$, $\sin B = 4/5$ and $\sin C = 1$ respectively.

In any triangle ABC, prove the following:

$$4. \frac{a-b}{a+b} = \frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}$$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{a}{\sin A} = k$$

So, $a = k \sin A$

Similarly, $b = k \sin B$

And $c = k \sin C$

We know,

$$a - b = k (\sin A - \sin B)$$

$$a + b = k (\sin A + \sin B)$$

Now let us consider LHS:

$$\begin{aligned} \frac{a - b}{a + b} &= \frac{k(\sin A - \sin B)}{k(\sin A + \sin B)} \\ &= \frac{(\sin A - \sin B)}{(\sin A + \sin B)} \dots (i) \end{aligned}$$

We know,

$$\sin A - \sin B = 2 \sin \frac{(A-B)}{2} \cos \frac{(A+B)}{2}$$

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2}$$

Substituting the above formulas in equation (i), we get

$$\frac{a - b}{a + b} = \frac{\left(2 \sin \left(\frac{A-B}{2}\right) \cos \left(\frac{A+B}{2}\right)\right)}{\left(2 \sin \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{2}\right)\right)}$$

Upon rearranging we get,

$$\begin{aligned} &= \frac{\left(\sin \left(\frac{A-B}{2}\right)\right)}{\left(\cos \left(\frac{A-B}{2}\right)\right)} \times \frac{\cos \left(\frac{A+B}{2}\right)}{\sin \left(\frac{A+B}{2}\right)} \\ &= \frac{\left(\tan \left(\frac{A-B}{2}\right)\right)}{1} \times \frac{1}{\tan \left(\frac{A+B}{2}\right)} \end{aligned}$$

$$= \frac{\left(\tan \left(\frac{A-B}{2}\right)\right)}{\left(\tan \left(\frac{A+B}{2}\right)\right)}$$

= RHS

Hence proved.

$$5. (a - b) \cos C/2 = C \sin (A - B)/2$$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{a}{\sin A} = k$$

$$\text{So, } a = k \sin A$$

$$\text{Similarly, } b = k \sin B$$

We know,

$$a - b = k (\sin A - \sin B) \dots (i)$$

Now let us consider LHS:

$$(a - b) \cos \frac{C}{2}$$

Substituting equation (i) in above equation, we get

$$(k(\sin A - \sin B)) \cos \frac{C}{2} \dots (ii)$$

We know,

$$\sin A - \sin B = 2 \sin (A-B)/2 \cos (A+B)/2$$

Substituting the above formulas in equation (ii), we get

$$\begin{aligned} (a - b) \cos \frac{C}{2} &= \left(k \left(2 \sin \left(\frac{A - B}{2} \right) \cos \left(\frac{A + B}{2} \right) \right) \right) \cos \frac{C}{2} \\ &= \left(k \left(2 \sin \left(\frac{A - B}{2} \right) \cos \left(\frac{A + B}{2} \right) \right) \right) \cos \frac{(\pi - (A + B))}{2} \\ &= \left(2k \sin \left(\frac{A - B}{2} \right) \cos \left(\frac{A + B}{2} \right) \right) \sin \left(\frac{A + B}{2} \right) \quad [\text{since, } \cos(\pi/2 - A) = \sin A] \end{aligned}$$

Upon rearranging we get,

$$= k \sin \left(\frac{A - B}{2} \right) \left(2 \sin \left(\frac{A + B}{2} \right) \cos \left(\frac{A + B}{2} \right) \right)$$

$$\text{We know, } \sin A = 2 \cos (A/2) \sin (A/2)$$

So the above equation becomes,

$$\begin{aligned} &= k \sin\left(\frac{A-B}{2}\right) (\sin(A+B)) \\ &= k \sin\left(\frac{A-B}{2}\right) (\sin(\pi-C)) \quad [\text{since, } \pi = A+B+C, \text{ where, } A+B = \pi-C] \\ &= k \sin(C) \sin\left(\frac{A-B}{2}\right) \quad [\text{since, } \sin(\pi-A) = \sin A] \end{aligned}$$

From the sine rule,

$$\frac{c}{\sin C} = k \Rightarrow c = k \sin C$$

So the above equation becomes,

$$= c \sin\left(\frac{A-B}{2}\right)$$

= RHS

Hence proved.

$$6. \frac{c}{a-b} = \frac{\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right)}{\tan\left(\frac{A}{2}\right) - \tan\left(\frac{B}{2}\right)}$$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{a}{\sin A} = k$$

So, $a = k \sin A$

Similarly, $b = k \sin B$

And $c = k \sin C \dots$ (i)

We know,

$$a - b = k (\sin A - \sin B) \dots$$
 (ii)

Now let us consider LHS:

$$\frac{c}{a - b}$$

Substituting equation (i) and (ii) in above equation, we get

$$\frac{k \sin C}{k(\sin A - \sin B)} = \frac{\sin C}{(\sin A - \sin B)} \dots$$
 (iii)

By applying half angle rule,

$$\sin C = 2 \sin \frac{C}{2} \cos \frac{C}{2} \dots$$
 (iv)

And we know,

$$\sin A - \sin B = 2 \sin \frac{(A-B)}{2} \cos \frac{(A+B)}{2} \dots$$
 (v)

Substituting the above equations (iv) and (v) in equation (iii), we get

$$\begin{aligned} \frac{c}{(a - b)} &= \frac{2 \sin \frac{C}{2} \cos \frac{C}{2}}{2 \sin \left(\frac{A - B}{2} \right) \cos \left(\frac{A + B}{2} \right)} \\ &= \frac{\sin \left(\frac{\pi - (A + B)}{2} \right) \cos \left(\frac{C}{2} \right)}{\sin \left(\frac{A - B}{2} \right) \cos \left(\frac{A + B}{2} \right)} \quad [\text{since, } \pi = A + B + C, \text{ where, } C = \pi - (A + B)] \end{aligned}$$

$$= \frac{\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{C}{2}\right)}{\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)} \quad [\text{since, } \sin(\pi/2 - A) = \cos A]$$

Upon simplification we get,

$$\begin{aligned} &= \frac{\cos\left(\frac{C}{2}\right)}{\sin\left(\frac{A-B}{2}\right)} \\ &= \frac{\cos\left(\frac{\pi - (A+B)}{2}\right)}{\sin\left(\frac{A-B}{2}\right)} \quad [\text{since, } \pi = A+B+C, \text{ where, } C = \pi - (A+B)] \\ &= \frac{\sin\left(\frac{(A+B)}{2}\right)}{\sin\left(\frac{A-B}{2}\right)} \quad \dots \text{(vi)} \quad [\text{since, } \cos(\pi/2 - A) = \sin A] \end{aligned}$$

We know,

$$\sin(A+B)/2 = \sin(A/2 + B/2) = \sin A/2 \cos B/2 + \cos A/2 \sin B/2$$

$$\sin(A-B)/2 = \sin(A/2 - B/2) = \sin A/2 \cos B/2 - \cos A/2 \sin B/2$$

Substituting the above equations in equation (vi) we get,

$$= \frac{\sin \frac{A}{2} \cos \frac{B}{2} + \cos\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right)}{\sin \frac{A}{2} \cos \frac{B}{2} - \cos\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right)}$$

Let us divide the numerator and denominator by $\cos A/2 \cos B/2$, we get

$$\begin{aligned} &\frac{\sin \frac{A}{2} \cos \frac{B}{2} + \cos\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right)}{\cos \frac{A}{2} \cos \frac{B}{2}} \\ &= \frac{\sin \frac{A}{2} \cos \frac{B}{2} - \cos\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right)}{\cos \frac{A}{2} \cos \frac{B}{2}} \\ &= \frac{\frac{\sin \frac{A}{2} \cos \frac{B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} + \frac{\cos\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right)}{\cos \frac{A}{2} \cos \frac{B}{2}}}{\frac{\sin \frac{A}{2} \cos \frac{B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} - \frac{\cos\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right)}{\cos \frac{A}{2} \cos \frac{B}{2}}} \end{aligned}$$

Upon simplification we get,

$$\begin{aligned} & \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} + \frac{\sin \left(\frac{B}{2}\right)}{\cos \frac{B}{2}} \\ &= \frac{\frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} - \frac{\sin \left(\frac{B}{2}\right)}{\cos \frac{B}{2}}}{\frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} - \frac{\sin \left(\frac{B}{2}\right)}{\cos \frac{B}{2}}} \\ &= \frac{\tan \frac{A}{2} + \tan \frac{B}{2}}{\tan \frac{A}{2} - \tan \frac{B}{2}} \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$7. \frac{c}{a+b} = \frac{1 - \tan \left(\frac{A}{2}\right) \tan \left(\frac{B}{2}\right)}{1 + \tan \left(\frac{A}{2}\right) \tan \left(\frac{B}{2}\right)}$$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{a}{\sin A} = k$$

So, $a = k \sin A$

Similarly, $b = k \sin B$

And $c = k \sin C$... (i)

We know,

$$a + b = k (\sin A + \sin B) \dots (ii)$$

Now let us consider LHS:

$$\frac{c}{a+b}$$

Substituting equation (i) and (ii) in above equation, we get

$$\frac{k \sin C}{k(\sin A + \sin B)} = \frac{\sin C}{(\sin A + \sin B)} \dots (iii)$$

By applying half angle rule,

$$\sin C = 2 \sin \frac{C}{2} \cos \frac{C}{2} \dots (iv)$$

And we know,

$$\sin A + \sin B = 2 \sin (A+B)/2 \cos (A-B)/2 \dots (v)$$

Substituting the above equations (iv) and (v) in equation (iii), we get

$$\begin{aligned} \frac{c}{(a+b)} &= \frac{2 \sin \frac{C}{2} \cos \frac{C}{2}}{2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right)} \\ &= \frac{\sin \left(\frac{\pi - (A+B)}{2} \right) \cos \left(\frac{\pi - (A+B)}{2} \right)}{\sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right)} \end{aligned}$$

[Since, $\pi = A+B+C$, where,

$$\begin{aligned} C &= \pi - (A+B) \\ &= \frac{\cos \left(\frac{A+B}{2} \right) \sin \left(\frac{A+B}{2} \right)}{\sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right)} \end{aligned}$$

[Since, $\sin (\pi/2 - A) = \cos A$, $\cos (\pi/2 - A) = \sin A$]

Upon simplification we get,

$$= \frac{\cos \left(\frac{(A+B)}{2} \right)}{\cos \left(\frac{(A-B)}{2} \right)} \dots (vi)$$

We know,

$$\cos (A+B)/2 = \cos (A/2 + B/2) = \cos A/2 \cos B/2 - \sin A/2 \sin B/2$$

$$\cos (A-B)/2 = \cos (A/2 - B/2) = \cos A/2 \cos B/2 + \sin A/2 \sin B/2$$

Substituting the above equations in equation (vi) we get,

$$= \frac{\cos \frac{A}{2} \cos \frac{B}{2} + \sin \left(\frac{A}{2} \right) \sin \left(\frac{B}{2} \right)}{\cos \frac{A}{2} \cos \frac{B}{2} - \sin \left(\frac{A}{2} \right) \sin \left(\frac{B}{2} \right)}$$

Let us divide the numerator and denominator by $\cos \frac{A}{2} \cos \frac{B}{2}$, we get

$$\begin{aligned} & \frac{\cos \frac{A}{2} \cos \frac{B}{2} + \sin \left(\frac{A}{2} \right) \sin \left(\frac{B}{2} \right)}{\cos \frac{A}{2} \cos \frac{B}{2}} \\ &= \frac{\frac{\cos \frac{A}{2} \cos \frac{B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} + \frac{\sin \left(\frac{A}{2} \right) \sin \left(\frac{B}{2} \right)}{\cos \frac{A}{2} \cos \frac{B}{2}}}{\frac{\cos \frac{A}{2} \cos \frac{B}{2} - \sin \left(\frac{A}{2} \right) \sin \left(\frac{B}{2} \right)}{\cos \frac{A}{2} \cos \frac{B}{2}}} \\ &= \frac{1 + \frac{\sin \left(\frac{A}{2} \right) \sin \left(\frac{B}{2} \right)}{\cos \frac{A}{2} \cos \frac{B}{2}}}{1 - \frac{\sin \left(\frac{A}{2} \right) \sin \left(\frac{B}{2} \right)}{\cos \frac{A}{2} \cos \frac{B}{2}}} \end{aligned}$$

Upon simplification we get,

$$\begin{aligned} & 1 + \frac{\sin \left(\frac{A}{2} \right) \sin \left(\frac{B}{2} \right)}{\cos \frac{A}{2} \cos \frac{B}{2}} \\ &= \frac{1 + \frac{\sin \left(\frac{A}{2} \right) \sin \left(\frac{B}{2} \right)}{\cos \frac{A}{2} \cos \frac{B}{2}}}{1 - \frac{\sin \left(\frac{A}{2} \right) \sin \left(\frac{B}{2} \right)}{\cos \frac{A}{2} \cos \frac{B}{2}}} \\ &= \frac{1 + \tan \frac{A}{2} \tan \frac{B}{2}}{1 - \tan \frac{A}{2} \tan \frac{B}{2}} \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$8. \frac{a+b}{c} = \frac{\cos \left(\frac{A-B}{2} \right)}{\sin \frac{C}{2}}$$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{a}{\sin A} = k$$

$$\text{So, } a = k \sin A$$

$$\text{Similarly, } b = k \sin B$$

$$\text{And } c = k \sin C \dots (i)$$

We know,

$$a + b = k (\sin A + \sin B) \dots (ii)$$

Now let us consider LHS:

$$\frac{a + b}{c}$$

Substituting equation (i) and (ii) in above equation, we get

$$\frac{k(\sin A + \sin B)}{k(\sin C)} = \frac{(\sin A + \sin B)}{(\sin C)} \dots (iii)$$

By applying half angle rule,

$$\sin C = 2 \sin \frac{C}{2} \cos \frac{C}{2} \dots (iv)$$

And we know,

$$\sin A + \sin B = 2 \sin \frac{(A+B)}{2} \cos \frac{(A-B)}{2} \dots (v)$$

Substituting the above equations (iv) and (v) in equation (iii), we get

$$\begin{aligned} \frac{a + b}{c} &= \frac{2 \sin \left(\frac{A + B}{2} \right) \cos \left(\frac{A - B}{2} \right)}{2 \sin \left(\frac{C}{2} \right) \cos \left(\frac{C}{2} \right)} \\ &= \frac{\sin \left(\frac{A + B}{2} \right) \cos \left(\frac{A - B}{2} \right)}{\sin \left(\frac{C}{2} \right) \cos \left(\frac{\pi - (A + B)}{2} \right)} \quad [\text{Since, } \pi = A + B + C, \text{ where, } C = \pi - (A + B)] \end{aligned}$$

$$= \frac{\sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right) \sin\left(\frac{A+B}{2}\right)} \quad [\text{Since, } \sin(\pi/2 - A) = \cos A, \cos(\pi/2 - A) = \sin A]$$

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Upon simplification we get,

$$= \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)}$$

= RHS

Hence proved.

$$9. \sin\left(\frac{B-C}{2}\right) = \frac{b-c}{a} \cos\frac{A}{2}$$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{c}{\sin C} = k$$

So, $c = k \sin C$

Similarly, $b = k \sin B$

We know,

$$b - c = k(\sin B - \sin C) \dots (i)$$

Now let us consider RHS:

$$\frac{b-c}{a} \cos\frac{A}{2}$$

Substituting equation (i) in above equation, we get

$$\frac{(k(\sin B - \sin C))}{k \sin A} \cos\frac{A}{2} = \frac{(\sin B - \sin C)}{\sin A} \cos\frac{A}{2} \dots (ii)$$

And we know,

$$\sin B - \sin C = 2 \sin\left(\frac{B-C}{2}\right) \cos\left(\frac{B+C}{2}\right) \dots (iii)$$

Substituting the above equation (iii) in equation (ii), we get

$$\frac{b-c}{a} \cos\frac{A}{2} = \frac{2 \sin\left(\frac{B-C}{2}\right) \cos\left(\frac{B+C}{2}\right)}{\sin A} \cos\left(\frac{\pi - (B+C)}{2}\right) \quad [\text{Since, } \pi = A+B+C, \text{ where, } C = \pi - (A+B)]$$

$$= \frac{2 \sin\left(\frac{B-C}{2}\right) \cos\left(\frac{B+C}{2}\right)}{\sin A} \sin\left(\frac{(B+C)}{2}\right) \quad [\text{Since, } \cos(\pi/2 - A)]$$

$$= \sin A]$$

Upon rearranging the above equation we get,

$$= \frac{\sin\left(\frac{B-C}{2}\right) \left(2 \sin\left(\frac{(B+C)}{2}\right) \cos\left(\frac{(B+C)}{2}\right)\right)}{\sin A}$$

We know $\sin A = 2 \cos(A/2) \sin(A/2)$

So,

$$\begin{aligned} &= \frac{\sin\left(\frac{B-C}{2}\right) (\sin(B+C))}{\sin A} \\ &= \frac{\sin\left(\frac{B-C}{2}\right) (\sin(\pi - A))}{\sin A} \quad [\text{Since, } \pi = A+B+C, \text{ where, } A+B = \pi-C] \\ &= \frac{\sin\left(\frac{B-C}{2}\right) \sin A}{\sin A} \quad [\text{Since, } \sin(\pi-A) = \sin A] \end{aligned}$$

Upon simplification we get,

$$\begin{aligned} &= \sin\left(\frac{B-C}{2}\right) \\ &= \text{LHS} \end{aligned}$$

Hence proved.

$$10. \frac{a^2 - c^2}{b^2} = \frac{\sin(A - C)}{\sin(A + C)}$$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{c}{\sin C} = k$$

So, $c = k \sin C$

Similarly, $a = k \sin A$

And $b = k \sin B$

So, $a - c = k(\sin A - \sin C) \dots (i)$

We know,

Now let us consider LHS:

$$\frac{a^2 - c^2}{b^2}$$

Substituting the values in the above equation, we get

$$\frac{(k \sin A)^2 - (k \sin C)^2}{(k \sin B)^2} = \frac{k^2(\sin^2 A - \sin^2 C)}{k^2 \sin^2 B} \dots (ii)$$

And we know,

$\sin^2 A - \sin^2 C = \sin(A + C) \sin(A - C) \dots (iii)$

Substituting the above equation (iii) in equation (ii), we get

$$\begin{aligned} \frac{a^2 - c^2}{b^2} &= \frac{\sin(A + C) \sin(A - C)}{\sin^2(\pi - (A + C))} \quad [\text{Since, } \pi = A + B + C, \text{ where, } C = \pi - (A + B)] \\ &= \frac{\sin(A + C) \sin(A - C)}{\sin^2((A + C))} \quad [\text{Since, } \sin(\pi - A) = \sin A] \\ &= \frac{\sin(A - C)}{\sin(A + C)} \\ &= \text{RHS} \end{aligned}$$

Hence proved.

$$11. b \sin B - c \sin C = a \sin(B - C)$$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{c}{\sin C} = k$$

So, $c = k \sin C$

Similarly, $a = k \sin A$

And $b = k \sin B$

We know,

Now let us consider LHS:

$$b \sin B - c \sin C$$

Substituting the values of b and c in the above equation, we get

$$k \sin B \sin B - k \sin C \sin C = k (\sin^2 B - \sin^2 C) \dots\dots\dots(i)$$

We know,

$$\sin^2 B - \sin^2 C = \sin (B + C) \sin (B - C),$$

Substituting the above values in equation (i), we get

$$k (\sin^2 B - \sin^2 C) = k (\sin (B + C) \sin (B - C)) \text{ [since, } \pi = A + B + C \Rightarrow B + C = \pi - A]$$

The above equation becomes,

$$= k (\sin (\pi - A) \sin (B - C)) \text{ [since, } \sin (\pi - \theta) = \sin \theta]$$

$$= k (\sin (A) \sin (B - C))$$

From sine rule, $a = k \sin A$, so the above equation becomes,

$$= a \sin (B - C)$$

$$= \text{RHS}$$

Hence proved.

$$\mathbf{12. a^2 \sin (B - C) = (b^2 - c^2) \sin A}$$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\frac{c}{\sin C} = k$$

So, $c = k \sin C$

Similarly, $a = k \sin A$

And $b = k \sin B$

We know,

Now let us consider RHS:

$$(b^2 - c^2) \sin A \dots$$

Substituting the values of b and c in the above equation, we get

$$\begin{aligned}(b^2 - c^2) \sin A &= [(k \sin B)^2 - (k \sin C)^2] \sin A \\ &= k^2 (\sin^2 B - \sin^2 C) \sin A \dots\dots\dots (i)\end{aligned}$$

We know,

$$\sin^2 B - \sin^2 C = \sin(B + C) \sin(B - C),$$

Substituting the above values in equation (i), we get

$$\begin{aligned}&= k^2 (\sin(B + C) \sin(B - C)) \sin A \text{ [since, } \pi = A + B + C \Rightarrow B + C = \pi - A\text{]} \\ &= k^2 (\sin(\pi - A) \sin(B - C)) \sin A \\ &= k^2 (\sin A) \sin(B - C) \sin A \text{ [since, } \sin(\pi - \theta) = \sin \theta\text{]}\end{aligned}$$

Rearranging the above equation we get

$$= (k \sin A) (\sin(B - C)) (k \sin A)$$

From sine rule, $a = k \sin A$, so the above equation becomes,

$$= a^2 \sin(B - C)$$

$$= \text{RHS}$$

Hence proved.

$$13. \frac{\sqrt{\sin A} - \sqrt{\sin B}}{\sqrt{\sin A} + \sqrt{\sin B}} = \frac{a + b - 2\sqrt{ab}}{a - b}$$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\sin A = \frac{a}{k}, \sin B = \frac{b}{k}, \sin C = \frac{c}{k}$$

Let us consider LHS,

$$\frac{\sqrt{\sin A} - \sqrt{\sin B}}{\sqrt{\sin A} + \sqrt{\sin B}}$$

Let us multiply and divide the above expression by $\frac{\sqrt{\sin A} - \sqrt{\sin B}}{\sqrt{\sin A} - \sqrt{\sin B}}$ we get,

$$\begin{aligned} \frac{\sqrt{\sin A} - \sqrt{\sin B}}{\sqrt{\sin A} + \sqrt{\sin B}} \times \frac{\sqrt{\sin A} - \sqrt{\sin B}}{\sqrt{\sin A} - \sqrt{\sin B}} &= \frac{(\sqrt{\sin A} - \sqrt{\sin B})^2}{(\sqrt{\sin A})^2 - (\sqrt{\sin B})^2} \\ &= \frac{(\sqrt{\sin A})^2 + (\sqrt{\sin B})^2 - (2\sqrt{\sin A} \times \sqrt{\sin B})}{\sin A - \sin B} \\ &= \frac{\sin A + \sin B - (2\sqrt{\sin A \times \sin B})}{\sin A - \sin B} \end{aligned}$$

Substituting the values of a and b from sine rule in the above equation, we get

$$\begin{aligned} &= \frac{\frac{a}{k} + \frac{b}{k} - \left(2\sqrt{\frac{a}{k} \times \frac{b}{k}}\right)}{\frac{a}{k} - \frac{b}{k}} \\ &= \frac{\frac{1}{k}(a + b - 2\sqrt{ab})}{\frac{1}{k}(a - b)} \\ &= \frac{a + b - 2\sqrt{ab}}{a - b} \end{aligned}$$

= RHS

Hence proved.

14. $a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B) = 0$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$a = k \sin A, b = k \sin B, c = k \sin C$$

Let us consider LHS:

$$a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B)$$

Substituting the values of a, b, c from sine rule in above equation, we get

$$a (\sin B - \sin C) + b (\sin C - \sin A) + c (\sin A - \sin B) = k \sin A (\sin B - \sin C) + k \sin B (\sin C - \sin A) + k \sin C (\sin A - \sin B)$$

$$= k \sin A \sin B - k \sin A \sin C + k \sin B \sin C - k \sin B \sin A + k \sin C \sin A - k \sin C \sin B$$

Upon simplification, we get

$$= 0$$

$$= \text{RHS}$$

Hence proved.

$$15. \frac{a^2 \sin(B - C)}{\sin A} + \frac{b^2 \sin(C - A)}{\sin B} + \frac{c^2 \sin(A - B)}{\sin C} = 0$$

Solution:

By using the sine rule we know,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$a = k \sin A, b = k \sin B, c = k \sin C$$

Let us consider LHS:

$$\frac{a^2 \sin(B - C)}{\sin A} + \frac{b^2 \sin(C - A)}{\sin B} + \frac{c^2 \sin(A - B)}{\sin C}$$

Substituting the values of a, b and c from sine rule in the above equation, we get

$$\begin{aligned} &= \frac{(k \sin A)^2 \sin(B - C)}{\sin A} + \frac{(k \sin B)^2 \sin(C - A)}{\sin B} + \frac{(k \sin C)^2 \sin(A - B)}{\sin C} \\ &= \frac{k^2 \sin^2 A \sin(B - C)}{\sin A} + \frac{k^2 \sin^2 B \sin(C - A)}{\sin B} + \frac{k^2 \sin^2 C \sin(A - B)}{\sin C} \end{aligned}$$

Upon simplification we get,

$$= k^2 [\sin A \sin(B - C) + \sin B \sin(C - A) + \sin C \sin(A - B)]$$

We know, $\sin(A - B) = \sin A \cos B - \cos A \sin B$

$$\sin(B - C) = \sin B \cos C - \cos B \sin C$$

$$\sin(C - A) = \sin C \cos A - \cos C \sin A$$

So the above equation becomes,

$$= k^2 [\sin A (\sin B \cos C - \cos B \sin C) + \sin B (\sin C \cos A - \cos C \sin A) + \sin C (\sin A \cos B - \cos A \sin B)]$$

$$= k^2 [\sin A \sin B \cos C - \sin A \cos B \sin C + \sin B \sin C \cos A - \sin B \cos C \sin A + \sin C \sin A \cos B - \sin C \cos A \sin B]$$

Upon simplification we get,

$$= 0$$

= RHS

Hence proved.

EXERCISE 10.2 PAGE NO: 10.25

In any $\triangle ABC$, prove the following:

1. In a $\triangle ABC$, if $a = 5$, $b = 6$ and $C = 60^\circ$, show that its area is $(15\sqrt{3})/2$ sq. units.

Solution:

Given:

In a $\triangle ABC$, $a = 5$, $b = 6$ and $C = 60^\circ$

By using the formula,

Area of $\triangle ABC = 1/2 ab \sin \theta$ where, a and b are the lengths of the sides of a triangle and θ is the angle between sides.

So,

$$\text{Area of } \triangle ABC = 1/2 ab \sin \theta$$

$$= 1/2 \times 5 \times 6 \times \sin 60^\circ$$

$$= 30/2 \times \sqrt{3}/2$$

$$= (15\sqrt{3})/2 \text{ sq. units}$$

Hence proved.

2. In a $\triangle ABC$, if $a = \sqrt{2}$, $b = \sqrt{3}$ and $c = \sqrt{5}$ show that its area is $1/2 \sqrt{6}$ sq. units.

Solution:

Given:

In a $\triangle ABC$, $a = \sqrt{2}$, $b = \sqrt{3}$ and $c = \sqrt{5}$

By using the formulas,

$$\text{We know, } \cos A = (b^2 + c^2 - a^2)/2bc$$

By substituting the values we get,

$$= [(\sqrt{3})^2 + (\sqrt{5})^2 - (\sqrt{2})^2] / [2 \times \sqrt{3} \times \sqrt{5}]$$

$$= 3/\sqrt{15}$$

$$\text{We know, Area of } \triangle ABC = 1/2 bc \sin A$$

To find $\sin A$:

$$\sin A = \sqrt{(1 - \cos^2 A)} \text{ [by using trigonometric identity]}$$

$$= \sqrt{(1 - (3/\sqrt{15})^2)}$$

$$= \sqrt{(1 - (9/15))}$$

$$= \sqrt{(6/15)}$$

Now,

$$\text{Area of } \triangle ABC = \frac{1}{2} bc \sin A$$

$$= \frac{1}{2} \times \sqrt{3} \times \sqrt{5} \times \sqrt{\frac{6}{15}}$$

$$= \frac{1}{2} \sqrt{6} \text{ sq. units}$$

Hence proved.

3. The sides of a triangle are $a = 4$, $b = 6$ and $c = 8$, show that: $8 \cos A + 16 \cos B + 4 \cos C = 17$.

Solution:

Given:

Sides of a triangle are $a = 4$, $b = 6$ and $c = 8$

By using the formulas,

$$\cos A = \frac{(b^2 + c^2 - a^2)}{2bc}$$

$$\cos B = \frac{(a^2 + c^2 - b^2)}{2ac}$$

$$\cos C = \frac{(a^2 + b^2 - c^2)}{2ab}$$

So now let us substitute the values of a , b and c we get,

$$\cos A = \frac{(b^2 + c^2 - a^2)}{2bc}$$

$$= \frac{(6^2 + 8^2 - 4^2)}{2 \times 6 \times 8}$$

$$= \frac{(36 + 64 - 16)}{96}$$

$$= \frac{84}{96}$$

$$= \frac{7}{8}$$

$$\cos B = \frac{(a^2 + c^2 - b^2)}{2ac}$$

$$= \frac{(4^2 + 8^2 - 6^2)}{2 \times 4 \times 8}$$

$$= \frac{(16 + 64 - 36)}{64}$$

$$= \frac{44}{64}$$

$$\cos C = \frac{(a^2 + b^2 - c^2)}{2ab}$$

$$= \frac{(4^2 + 6^2 - 8^2)}{2 \times 4 \times 6}$$

$$= \frac{(16 + 36 - 64)}{48}$$

$$= -\frac{12}{48}$$

$$= -\frac{1}{4}$$

Now considering LHS:

$$8 \cos A + 16 \cos B + 4 \cos C = 8 \times \frac{7}{8} + 16 \times \frac{44}{64} + 4 \times \left(-\frac{1}{4}\right)$$

$$= 7 + 11 - 1$$

$$= 17$$

Hence proved.

4. In a $\triangle ABC$, if $a = 18$, $b = 24$, $c = 30$, find $\cos A$, $\cos B$ and $\cos C$

Solution:

Given:

Sides of a triangle are $a = 18$, $b = 24$ and $c = 30$

By using the formulas,

$$\cos A = (b^2 + c^2 - a^2)/2bc$$

$$\cos B = (a^2 + c^2 - b^2)/2ac$$

$$\cos C = (a^2 + b^2 - c^2)/2ab$$

So now let us substitute the values of a , b and c we get,

$$\cos A = (b^2 + c^2 - a^2)/2bc$$

$$= (24^2 + 30^2 - 18^2)/2 \times 24 \times 30$$

$$= 1152/1440$$

$$= 4/5$$

$$\cos B = (a^2 + c^2 - b^2)/2ac$$

$$= (18^2 + 30^2 - 24^2)/2 \times 18 \times 30$$

$$= 648/1080$$

$$= 3/5$$

$$\cos C = (a^2 + b^2 - c^2)/2ab$$

$$= (18^2 + 24^2 - 30^2)/2 \times 18 \times 24$$

$$= 0/864$$

$$= 0$$

$$\therefore \cos A = 4/5, \cos B = 3/5, \cos C = 0$$

5. For any $\triangle ABC$, show that $b(c \cos A - a \cos C) = c^2 - a^2$

Solution:

Let us consider LHS:

$$b(c \cos A - a \cos C)$$

As LHS contain $bc \cos A$ and $ab \cos C$ which can be obtained from cosine formulae.

From cosine formula we have:

$$\cos A = (b^2 + c^2 - a^2)/2bc$$

$$bc \cos A = (b^2 + c^2 - a^2)/2 \dots (i)$$

$$\cos C = (a^2 + b^2 - c^2)/2ab$$

$$ab \cos C = (a^2 + b^2 - c^2)/2 \dots (ii)$$

Now let us subtract equation (i) and (ii) we get,

$$\begin{aligned}bc \cos A - ab \cos C &= (b^2 + c^2 - a^2)/2 - (a^2 + b^2 - c^2)/2 \\&= c^2 - a^2\end{aligned}$$

$$\therefore b(c \cos A - a \cos C) = c^2 - a^2$$

Hence proved.

6. For any ΔABC show that $c(a \cos B - b \cos A) = a^2 - b^2$

Solution:

Let us consider LHS:

$$c(a \cos B - b \cos A)$$

As LHS contain $ca \cos B$ and $cb \cos A$ which can be obtained from cosine formulae.

From cosine formula we have:

$$\cos A = (b^2 + c^2 - a^2)/2bc$$

$$bc \cos A = (b^2 + c^2 - a^2)/2 \dots (i)$$

$$\cos B = (a^2 + c^2 - b^2)/2ac$$

$$ac \cos B = (a^2 + c^2 - b^2)/2 \dots (ii)$$

Now let us subtract equation (ii) from (i) we get,

$$\begin{aligned}ac \cos B - bc \cos A &= (a^2 + c^2 - b^2)/2 - (b^2 + c^2 - a^2)/2 \\&= a^2 - b^2\end{aligned}$$

$$\therefore c(a \cos B - b \cos A) = a^2 - b^2$$

Hence proved.

7. For any ΔABC show that

$$2(bc \cos A + ca \cos B + ab \cos C) = a^2 + b^2 + c^2$$

Solution:

Let us consider LHS:

$$2(bc \cos A + ca \cos B + ab \cos C)$$

As LHS contain $2ca \cos B$, $2ab \cos C$ and $2cb \cos A$, which can be obtained from cosine formulae.

From cosine formula we have:

$$\cos A = (b^2 + c^2 - a^2)/2bc$$

$$2bc \cos A = (b^2 + c^2 - a^2) \dots (i)$$

$$\cos B = (a^2 + c^2 - b^2)/2ac$$

$$2ac \cos B = (a^2 + c^2 - b^2) \dots (ii)$$

$$\cos C = (a^2 + b^2 - c^2)/2ab$$

$$2ab \cos C = (a^2 + b^2 - c^2) \dots (iii)$$

Now let us add equation (i), (ii) and (iii) we get,

$$2bc \cos A + 2ac \cos B + 2ab \cos C = (b^2 + c^2 - a^2) + (a^2 + c^2 - b^2) + (a^2 + b^2 - c^2)$$

Upon simplification we get,

$$= c^2 + b^2 + a^2$$

$$2(bc \cos A + ac \cos B + ab \cos C) = a^2 + b^2 + c^2$$

Hence proved.

8. For any ΔABC show that

$$(c^2 - a^2 + b^2) \tan A = (a^2 - b^2 + c^2) \tan B = (b^2 - c^2 + a^2) \tan C$$

Solution:

Let us consider LHS:

$$(c^2 - a^2 + b^2), (a^2 - b^2 + c^2), (b^2 - c^2 + a^2)$$

We know sine rule in ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

As LHS contain $(c^2 - a^2 + b^2)$, $(a^2 - b^2 + c^2)$ and $(b^2 - c^2 + a^2)$, which can be obtained from cosine formulae.

From cosine formula we have:

$$\cos A = (b^2 + c^2 - a^2)/2bc$$

$$2bc \cos A = (b^2 + c^2 - a^2)$$

Let us multiply both the sides by $\tan A$ we get,

$$2bc \cos A \tan A = (b^2 + c^2 - a^2) \tan A$$

$$2bc \cos A (\sin A / \cos A) = (b^2 + c^2 - a^2) \tan A$$

$$2bc \sin A = (b^2 + c^2 - a^2) \tan A \dots (i)$$

$$\cos B = (a^2 + c^2 - b^2)/2ac$$

$$2ac \cos B = (a^2 + c^2 - b^2)$$

Let us multiply both the sides by $\tan B$ we get,

$$2ac \cos B \tan B = (a^2 + c^2 - b^2) \tan B$$

$$2ac \cos B (\sin B / \cos B) = (a^2 + c^2 - b^2) \tan B$$

$$2ac \sin B = (a^2 + c^2 - b^2) \tan B \dots (ii)$$

$$\cos C = (a^2 + b^2 - c^2)/2ab$$

$$2ab \cos C = (a^2 + b^2 - c^2)$$

Let us multiply both the sides by $\tan C$ we get,

$$2ab \cos C \tan C = (a^2 + b^2 - c^2) \tan C$$

$$2ab \cos C (\sin C / \cos C) = (a^2 + b^2 - c^2) \tan C$$

$$2ab \sin C = (a^2 + b^2 - c^2) \tan C \dots (iii)$$

As we are observing that sin terms are being involved so let's use sine formula.

From sine formula we have,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Let us multiply abc to each of the expression we get,

$$\frac{abc \sin A}{a} = \frac{abc \sin B}{b} = \frac{abc \sin C}{c}$$

$$bc \sin A = ac \sin B = ab \sin C$$

$$2bc \sin A = 2ac \sin B = 2ab \sin C$$

∴ From equation (i), (ii) and (iii) we have,

$$(c^2 - a^2 + b^2) \tan A = (a^2 - b^2 + c^2) \tan B = (b^2 - c^2 + a^2) \tan C$$

Hence proved.

9. For any Δ ABC show that:

$$\frac{c - b \cos A}{b - c \cos A} = \frac{\cos B}{\cos C}$$

Solution:

Let us consider LHS:

$$\frac{c - b \cos A}{b - c \cos A}$$

We can observe that we can get terms $c - b \cos A$ and $b - c \cos A$ from projection formula

From projection formula we get,

$$c = a \cos B + b \cos A$$

$$c - b \cos A = a \cos B \dots (i)$$

And,

$$b = c \cos A + a \cos C$$

$$b - c \cos A = a \cos C \dots (ii)$$

Dividing equation (i) by (ii), we get,

$$\begin{aligned} \frac{c - b \cos A}{b - c \cos A} &= \frac{a \cos B}{a \cos C} \\ &= \frac{\cos B}{\cos C} \end{aligned}$$

= RHS

Hence proved.