Date: 29/10/2023

Max. Mark. 102

Number of Questions: 6

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SOLUTIONS

for

Regional Mathematical Olympiad (2023)

INSTRUCTIONS TO CANDIDATES

1. Calculators (in any form) and protractors are not allowed.
2. Rulers and compasses are allowed.
3. All questions carry equal marks. Maximum marks: 102.
4. No marks will be awarded for stating an answer without justification.
5. Answer all the questions.
6. Answer to each question should start on a new page. Clearly, indicate the question number.

RMO

1. Let N be the set of all positive integers and $S = \{(a, b, c, d) \in N^4 : a^2 + b^2 + c^2 = d^2\}$. Find the largest positive integer m such that m divides $abcd$ for all $(a, b, c, d) \in S$.

Sol. $a^2 + b^2 + c^2 = d^2$

Let a, b, c are odd $\Rightarrow$ d is odd

$a^2, b^2, c^2 \rightarrow 4k + 1$

$a^2 + b^2 + c^2 \rightarrow 4k + 3$

But d^2 can't be of form $4k + 3$

Let any one is even without loss of generality lets assume a is even

$a^2 + b^2 + c^2 \rightarrow$ even $\Rightarrow$ d is even

$\Rightarrow$ $abcd$ is divisible by 4.

Let any two numbers be even

$\Rightarrow$ $abcd$ is divisible by 4

$\Rightarrow$ $abcd$ must be divisible by 4

Now, let $a, b, c \in \{3k + 1, 3k + 2\}$

$a^2, b^2, c^2 \in \{3k + 1\}$

$a^2 + b^2 + c^2 \in 3m$

$\Rightarrow$ $abcd$ is divisible by 3

$\Rightarrow$ Largest positive integer = 12

For example (a, b, c, d) = (1, 2, 2, 3), (2, 3, 6, 7), …

But all should be divided by 12, then largest positive integer is 12.

2. Let ω be a semicircle with AB as the bounding diameter and let CD be a variable chord of the semicircle of constant length such that C, D lie in the interior of the arc AB. Let E be a point on the diameter AB such that CE and DE are equally inclined to the line AB. Prove that

(a) The measure of $\angle CED$ is a constant

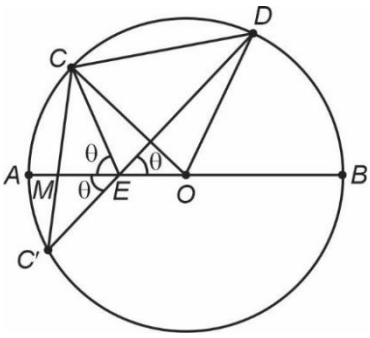
(b) The circumcircle of triangle CED passes through a fixed point.

Sol. Let $\angle CEA = \angle DEB = \theta$ (given)

$\Rightarrow$ $\angle CED = 180° - 2\theta$

Extend CE so that it meets the circle at point C'

$\Rightarrow$ $\angle MEC' = \theta$



Therefore, C and C' are mirror image of each other

$\Rightarrow \quad \angle EMC' = 90°$

$\Rightarrow \quad \angle MC'E = 90° - \theta$

Let O be the center of the circle, the angle made by chord CD at the centre is $2(90° - \theta) = 180° - 2\theta$

$\angle CED = \angle COD = 180° - 2\theta$

Therefore $CEOD$ are concyclic

Now in ΔCOD

$OD = OC = r$

$CD = 2r\sin(90° - \theta)$

$$\sin(90° - \theta) = \frac{l}{2r}$$

$\Rightarrow$ $90° - \theta$ is constant as l and r both are constant

$\Rightarrow$ θ is constant, $\angle CED = 180° - 2\theta$ is also constant

Now as we know that $CEOD$ are concyclic. The circumcircle of ΔCED will always pass through point O which is a fixed point.

3. For any natural number n, expressed in base 10, let $s(n)$ denote the sum of all its digits. Find all natural numbers m and n such that $m < n$ and $(s(n))^2 = m$ and $(s(m))^2 = n$.

Sol. $(s(n))^2 = m$ and $(s(m))^2 = n$

$m < n$

$\Rightarrow$ m and n are perfect square

Claim

"n and m cannot be more than 3 digit number"

If n is 4 digit number

$(s(n))_{max} = 36$

$m_{max} = 1296$

$\Rightarrow \quad m \le 1296$

$(s(m))_{max} = 19$ when $m = 676$

$\Rightarrow \quad n \le 361$

But this contradicts the fact that n is 4 digit number

Now lets consider n to be a 3 digit number

$(s(n))_{max} = 19$ when $m = 676$

$\Rightarrow \quad m \le 361$

Now checking

	10	11	12	**13**	14	15	**16**	17	18	19
m	100	121	144	**169**	196	225	**256**	289	324	361
$n = (s(m))^2$	1	4	81	**256**	256	81	**169**	361	81	100
$(s(n))^2$	1	16	81	**169**	169	81	**256**	100	81	1

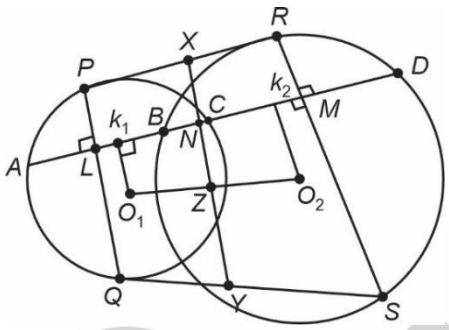
But $m < n$

$\Rightarrow \quad m = 169, n = 256$

Therefore, only possible pair is $(169, 256) \equiv (m, n)$

4. Let Ω_1, Ω_2 be two intersecting circles with centres O_1, O_2 respectively. Let l be a line that intersects Ω_1 at points A, C and Ω_2 at points B, D such that A, B, C, D are collinear in that order. Let the perpendicular bisector of segment AB intersect Ω_1 at points P, Q; and the perpendicular bisector of segment CD intersect Ω_2 at points R, S such that P, R are on the same side l. Prove that the midpoints of PR, QS and O_1O_2 are collinear.

Sol. Let the midpoint of PR, QS and O_1O_2 be X, Y and Z respectively foot of perpendiculars from points O_1 and O_2 onto the line $ABCD$ be k_1 and k_2 respectively. Now the line PQ and RS will be parallel as both are perpendicular to the line AD, therefore $PQRS$ is a trapezium let L and M be the intersections of PQ and RS with AD. As X and Y are midpoints, so XY will bisect LM by equal intercept theorem.



Now $O_1k_1k_2O_2$ will be a trapezium and let line XY meets k_1k_2 at N and O_1O_2 at Z.

To prove

$k_1N = Nk_2$

$LN - k_1L = MN - k_2M$ (as N is the mid-point of LM)

$k_1L = k_2M$

Let $AB = 2\alpha$, $BC = 2\beta$, $CD = 2\gamma$

$$AL = \frac{AB}{2} = \alpha$$

$$Ak_1 = \frac{AC}{2} = \alpha + \beta$$

$k_1L = Ak_1 - AL = \beta$

$$DM = \frac{CD}{2} = \gamma$$

$$Dk_2 = \frac{DB}{2} = \gamma + \beta$$

$k_2M = Dk_2 - DM = \beta$

$k_1L = k_2M = \beta$ (hence proved)

As N is the mid-point of k_1k_2, therefore Z will be mid-point of O_1O_2 by equal intercept theorem. So the points X, Y and Z will be collinear.

5. Let $n > k > 1$ be positive integers. Determine all positive real numbers a_1, a_2, …, a_n which satisfy

$$\sum_{i=1}^{n}\sqrt{\frac{ka_i^k}{(k-1)a_i^k+1}} = \sum_{i=1}^{n} a_i = n$$

Sol. $n > k > 1$ and $n, k \in I^+$ and $a_i \in$ positive real numbers

$$\sum_{i=1}^{n}\sqrt{\frac{ka_i^k}{(k-1)a_i^k+1}} = \sum_{i=1}^{n} a_i = n$$

By AM-GM inequality

$$\frac{\underbrace{a_i^k + a_i^k + a_i^k + \ldots + a_i^k}_{k-1} + 1}{k} \geq \left(\underbrace{a_i^k . a_i^k \ldots a_i^k}_{k-1} . 1\right)^{\frac{1}{k}}$$

$$\Rightarrow \quad (k-1)a_i^k + 1 \geq k(a_i)^{k-1}$$

Now

$$\sum_{i=1}^{n} \sqrt{\frac{ka_i^k}{(k-1)a_i^k + 1}} \leq \sum_{i=1}^{n} \sqrt{\frac{ka_i^k}{ka_i^{k-1}}}$$

$$\leq \sum_{i=1}^{n} \sqrt{a_i}$$

$$\Rightarrow \quad n = \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} \sqrt{a_i}$$

Now applying AM ≤ RMS

$$\frac{\sum_{i=1}^{n} \sqrt{a_i}}{n} \leq \sqrt{\frac{\sum_{i=1}^{n} a_i}{n}} = 1$$

$$\Rightarrow \quad \sum_{i=1}^{n} \sqrt{a_i} \leq n$$

$$\Rightarrow \quad \sum_{i=1}^{n} \sqrt{\frac{ka_i^k}{(k-1)a_i^k + 1}} \leq n$$

But given that $\sum_{i=1}^{n} \sqrt{\frac{ka_i^k}{(k-1)a_i^k + 1}} = n$

From inequality we will get

$$\Rightarrow \quad a_i = 1$$

6. Consider a set of 16 points arranged in a 4 × 4 square grid formation. Prove that if any 7 of these points are coloured blue, then there exists an isosceles right-angled triangle whose vertices are all blue.

Sol. Consider 4 × 4 grid with unit distance then

Case-I: Only 2 rows are used

All 7 points are in any two rows, then it is easy to find an isosceles right angled triangle as shown, in figure 6.1. Here $A_1C_1A_3$ is such triangle. $A_1A_3C_3$ is another such pairs of point to form isosceles right angled triangle with dimension $\left(2, 2, 2\sqrt{2}\right)$

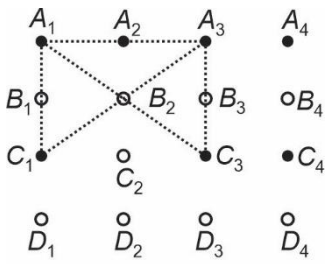


Figure 6.1

Case-II: All four rows are used

By Pigeonhole principle; at least three rows contain 2 points and 7th point can be in the remaining row.

Let's take a symmetric case; figure 6.2

Now, take point at D_4 then $A_1A_4D_4$ makes the required triangle with dimension $\left(4, 4, 4\sqrt{2}\right)$.

Similarly For $D_3 \rightarrow B_3C_4D_3$ will form such triangle with dimension $\left(\sqrt{2}, \sqrt{2}, 2\right)$

For $D_2 \rightarrow B_2C_1D_2$ forms required triangle with dimension $\left(\sqrt{2}, \sqrt{2}, 2\right)$

For $D_1 \rightarrow D_1A_1A_4$ forms required triangle with dimension $\left(4, 4, 4\sqrt{2}\right)$

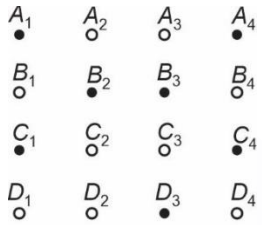


Figure 6.2

Let's take an asymmetric case of distribution of points as shown in figure 6.3

For point at D_1, $D_1A_1A_4$ forms isosceles right angled triangle with dimension $\left(4, 4, 4\sqrt{2}\right)$

For $D_2 \rightarrow D_2C_1B_2$ forms required triangle $\left(\sqrt{2}, \sqrt{2}, 2\right)$

For $D_3 \rightarrow D_3B_2A_4$ forms required triangle with dimension $\left(\sqrt{5}, \sqrt{5}, \sqrt{10}\right)$

For $D_4 \rightarrow D_4A_4A_1$ forms required triangle with dimension $\left(4, 4, 4\sqrt{2}\right)$

Or easily $A_1B_1B_2$ forms such right angled isosceles triangle.

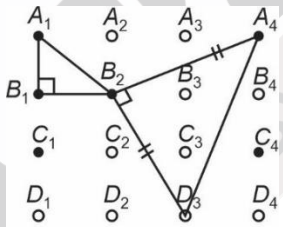


Figure 6.3

Case-III:

When only three rows are used

⇒ Again by Piegonhole principle each row will contain at least two points

⇒ Lets consider such colouring

For $D_3 \rightarrow D_3D_1B_1$ forms such triangle.

For $D_4 \rightarrow D_2D4B_4$ forms such required triangle.

Hence, an isosceles right angled triangle must exist in each case.

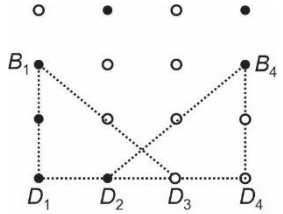


Figure 6.4

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